Problem 7.5

\[ I = \frac{E}{r + R}; \quad P = I^2 R = \frac{E^2 R}{(r + R)^2}; \quad \frac{dP}{dR} = e^2 \left[ \frac{1}{(r + R)^2} - \frac{2R}{(r + R)^3} \right] = 0 \Rightarrow r + R = 2R \Rightarrow R = r. \]

Problem 7.6

\[ E = \oint \mathbf{E} \cdot d\mathbf{l} = \text{[zero]} \] for all electrostatic fields. It looks as though \( E = \oint \mathbf{E} \cdot d\mathbf{l} = (\sigma / \varepsilon_0) h \), as would indeed be the case if the field were really just \( \sigma / \varepsilon_0 \) inside and zero outside. But in fact there is always a "fringing field" at the edges (Fig. 4.31), and this is evidently just right to kill off the contribution from the left end of the loop. The current is \( \text{[zero]} \).

Problem 7.7

(a) \( E = -\frac{d\Phi}{dt} = -Bl^2 \frac{dx}{dt} = -Blv; \quad E = IR \Rightarrow \boxed{I = \frac{Blv}{R}}. \) (Never mind the minus sign—it just tells you the direction of flow: \( \mathbf{v} \times \mathbf{B} \) is upward, in the bar, so downward through the resistor.)

(b) \( F = RB = \frac{B^2 l^2 v}{R} \) to the left.

(c) \( F = ma = m \frac{dv}{dt} = -\frac{B^2 l^2}{R} v \Rightarrow \frac{dv}{dt} = -\left( \frac{B^2 l^2}{Rm} \right) v \Rightarrow v = v_0 e^{-\alpha t}, \]

(d) The energy goes into heat in the resistor. The power delivered to resistor is \( I^2 R \), so

\[ \frac{dW}{dt} = I^2 R = \frac{B^2 l^2 v^2}{R} = \frac{B^2 l^2}{mR} v_0^2 e^{-2\alpha t}, \text{ where } \alpha = \frac{B^2 l^2}{mR}; \quad \frac{dW}{dt} = \alpha m v_0^2 e^{-2\alpha t}. \]

The total energy delivered to the resistor is

\[ W = \alpha m v_0^2 \int_0^\infty e^{-2\alpha t} dt = \alpha m v_0^2 \frac{e^{-2\alpha t}}{-2\alpha} \bigg|_0^\infty = \alpha m v_0^2 \frac{1}{2\alpha} = \frac{1}{2} m v_0^2. \vspace{1cm} \]

Problem 7.15

In the quasistatic approximation, \( \mathbf{B} = \begin{cases} \mu_0 I \frac{z}{s}, & (s < a); \\ 0, & (s > a). \end{cases} \)

Inside: for an "amperian loop" of radius \( s < a \),

\[ \Phi = B \pi s^2 = \mu_0 n I \pi s^2; \quad \oint \mathbf{E} \cdot d\mathbf{l} = E 2\pi s = -\frac{d\Phi}{dt} = -\mu_0 n \pi s^2 \frac{dI}{dt}; \quad \mathbf{E} = -\frac{\mu_0 n \pi s^2}{2} \frac{dI}{dt}. \]

Outside: for an "amperian loop" of radius \( s > a \):

\[ \Phi = B \pi a^2 = \mu_0 n I \pi a^2; \quad E 2\pi s = -\mu_0 n \pi a^2 \frac{dI}{dt}; \quad \mathbf{E} = -\frac{\mu_0 n a^2}{2s} \frac{dI}{dt}. \]

Problem 7.20

(a) From Eq. 5.38, the field (on the axis) is \( \mathbf{B} = \frac{\mu_0 I \frac{a^2}{2} \mathbf{r}}{(b^2 + z^2)^{3/2}} \), so the flux through the little loop (area \( \pi a^2 \))

\[ \Phi = -\frac{\mu_0 I \pi a^2 b^2}{2(b^2 + z^2)^{3/2}}. \]

(b) The field (Eq. 5.86) is \( \mathbf{B} = \frac{m r^2}{2(\cos \theta \hat{r} + \sin \theta \hat{\theta})} \), where \( m = I \pi a^2 \). Integrating over the spherical "cap" (bounded by the big loop and centered at the little loop):

\[ \Phi = \int \mathbf{B} \cdot d\mathbf{a} = \frac{\mu_0 I \pi a^2}{4\pi} \int (2 \cos \theta) (r^2 \sin \theta \, d\theta \, d\phi) = \frac{\mu_0 I \pi a^2}{2r} \int_0^{2\pi} \cos \theta \sin \theta \, d\theta \int_0^\theta \]
Problem 7.49

Initially, \( \frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2} \Rightarrow T = \frac{1}{2}mv^2 = \frac{1}{2} \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r} \). After the magnetic field is on, the electron circles in a new orbit, of radius \( r_1 \) and velocity \( v_1 \):

\[
\frac{mv_1^2}{r_1} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r_1^2} + qv_1B \Rightarrow T_1 = \frac{1}{2}mv_1^2 = \frac{1}{2} \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r_1} + \frac{1}{2}qv_1r_1B.
\]

But \( r_1 = r + dr \), so \( (r_1)^{-1} = (1 + \frac{dr}{r})^{-1} \equiv (1 - \frac{dr}{r})^{-1} \), while \( v_1 = v + dv \), \( B = dB \). To first order, then,

\[
T_1 = \frac{1}{2} \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r} \left( 1 - \frac{dr}{r} \right) + \frac{1}{2}q(vr)dB, \text{ and hence } dT = T_1 - T = \frac{qvr}{2}dB - \frac{1}{2} \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}dr.
\]

Now, the induced electric field is \( E = \frac{r dB}{dt} \) (Ex. 7.7), so \( m\frac{dv}{dt} = qE = \frac{q}{2}\frac{dB}{dt} \), or \( m\frac{dv}{dt} = \frac{q}{2}\frac{dB}{dt} \). The increase in kinetic energy is therefore \( dT = d(\frac{1}{2}mv^2) = mv dv = \frac{qvr}{2}dB \). Comparing the two expressions, I conclude that \( dr = 0 \). qed

Problem 7.26

(a) \( W = \frac{1}{2}LI^2 \). \( L = \mu_0n^2\pi R^2 l \) (Prob. 7.22) \[
W = \frac{1}{2} \mu_0n^2\pi R^2 l^2.
\]

(b) \( W = \frac{1}{2} \oint (A \cdot I) dl \). \( A = (\mu_0nI/2)R\hat{\phi} \), at the surface (Eq. 5.70 or 5.71). So \( W_1 = \frac{1}{2} \mu_0nI/2 \pi R \), for one turn. There are \( nl \) such turns in length \( l \), so \( W = \frac{1}{2} \mu_0n^2\pi R^2 l^2 \).

(c) \( W = \frac{1}{2\mu_0} \int B^2 \cdot dr \). \( B = \mu_0nI \), inside, and zero outside; \( \int dr = \pi R^2 l \), so \( W = \frac{1}{2\mu_0} \mu_0^2n^2\pi R^2 l = \frac{1}{2} \mu_0^2n^2\pi R^2 l^2 \).

(d) \( W = \frac{1}{2\mu_0} \left[ \int B^2 dr - \oint (A \times B) \cdot da \right] \). This time \( \int B^2 dr = \mu_0^2n^2\pi(R^2 - a^2)l \). Meanwhile, \( A \times B = 0 \) outside (at \( s = b \)). Inside, \( A = \frac{\mu_0nI}{2}a\hat{\phi} \) (at \( s = a \)), while \( B = \mu_0nI\hat{\phi} \).

\[
A \times B = \frac{1}{2} \mu_0^2n^2\pi a^2(\hat{\phi} \times \hat{\phi}) \quad \text{points inward ("out" of the volume)}
\]

\[
\oint (A \times B) \cdot da = \int \left( \frac{1}{2} \mu_0^2n^2\pi a^2 \right) \cdot a d\phi = -\frac{1}{2} \mu_0^2n^2\pi a^2 2\pi l.
\]

\[
W = \frac{1}{2\mu_0} \left[ \mu_0^2n^2\pi(R^2 - a^2)l + \mu_0^2n^2\pi a^2l^2 \right] = \frac{1}{2} \mu_0n^2\pi R^2 l^2. \quad \checkmark
\]