Universal Gap Fluctuations in the Superconductor Proximity Effect

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Random-matrix theory is used to study the mesoscopic fluctuations of the excitation gap in a metal grain or quantum dot induced by the proximity to a superconductor. We propose that the probability distribution of the gap is a universal function in rescaled units. Our analytical prediction for the gap distribution agrees well with exact diagonalization of a model Hamiltonian.

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A normal metal in the proximity of a superconductor acquires characteristics that are typical of the superconducting state [1]. One of those characteristics is that the quasiparticle density of states vanishes at the Fermi energy. This superconductor proximity effect is most pronounced in a confined geometry, such as a thin metal film or metal grain, or a semiconductor quantum dot. In that case, provided the scattering in the normal metal is chaotic, no excitations exist within an energy gap $E_g \sim \hbar/\tau$, where τ is the typical time between collisions with the superconductor [2–7].

If the coupling to the superconductor is weak (as for the point contact coupling of Fig. 1), the functional form of the density of states becomes independent of microscopic properties of the normal metal, such as the shape, dimensionality, or mean free path. Weak coupling means that τ is much bigger than the time $\tau_{\rm erg}$ needed for ergodic exploration of the phase space in the normal region [8]. For a point contact with $N \gg 1$ propagating modes at the Fermi level $\varepsilon = 0$, the density of states has a square-root singularity at the excitation gap [4],

$$\rho_{\rm mf}(\varepsilon) = \frac{1}{\pi}\sqrt{\frac{\varepsilon - E_g}{\Delta_g^3}}. \tag{1}$$

For a ballistic point contact and in the absence of a magnetic field, $E_g = cN\delta$ is the mean-field energy gap and $\Delta_g = c' N^{1/3}\delta$, where $c = 0.048$ and $c' = 0.068$ are numerical constants and δ is the mean level spacing in the normal metal when it is decoupled from the superconductor.

Equation (1) was obtained in a self-consistent diagrammatic perturbation theory that uses $\tau\delta/\hbar \sim N^{-1}$ as a small parameter. Such a mean-field theory provides a smoothed density of states for which energies can be resolved only on the scale of the rate $\hbar/\tau \sim N\delta$ between collisions with the superconductor, not on smaller energy scales, and is unable to deal with mesoscopic sample-to-sample fluctuations of the excitation gap. Mesoscopic fluctuations arise, e.g., upon varying the shape of a quantum dot or the impurity configuration in a metal grain. The lowest excited state ε_1 fluctuates from sample to sample around the mean-field value E_g, with a probability distribution $P(\varepsilon_1)$. It is the purpose of this paper to go beyond mean-field theory and to study the mesoscopic fluctuations of the excitation spectrum close to E_g. Our main result is that the gap distribution $P(\varepsilon_1)$ is a universal function of the rescaled energy $x = (\varepsilon_1 - E_g)/\Delta_g$, in a broad range $|x| \ll N^{2/3}$, where Δ_g is defined in terms of the mean-field density of states (1). The Fermi level itself ($\varepsilon = 0$) falls outside this range, which is why the universal gap distribution was not found in a recent related study [9]. Our main findings are illustrated in Fig. 2. Note that the width of the gap distribution $\Delta_g \sim E_g^{1/3}\delta^{2/3}$ is parametrically smaller than the gap size E_g but bigger than the mean level spacing δ in the dot.

Also note that, in terms of the rescaled variable x, the mean-field density $\rho_{\rm mf}$ is already universal, $\rho_{\rm mf}(x) = \pi^{-1}x^{1/2}$, though $\rho_{\rm mf}$ is different from the true ensemble averaged density of states $\langle\rho\rangle$; see Fig. 2. The difference could arise, because the mean-field theory is unable to resolve the density of states on the energy scale Δ_g.

We first consider the gap distribution in the absence of a magnetic field and then include a time-reversal symmetry

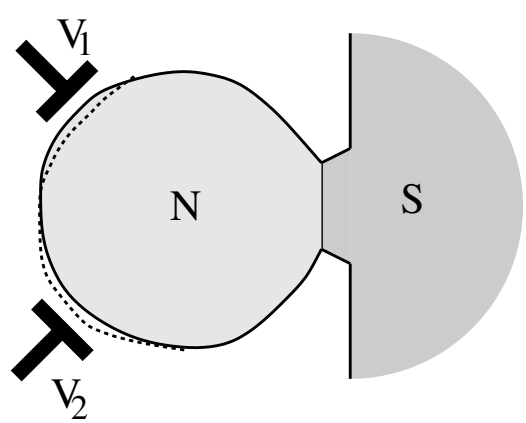

FIG. 1. A quantum dot (N) connected to a superconductor (S). The voltages on the gates V_1 and V_2 change the shape of the dot. Different values of the applied voltages create different samples within the same ensemble.

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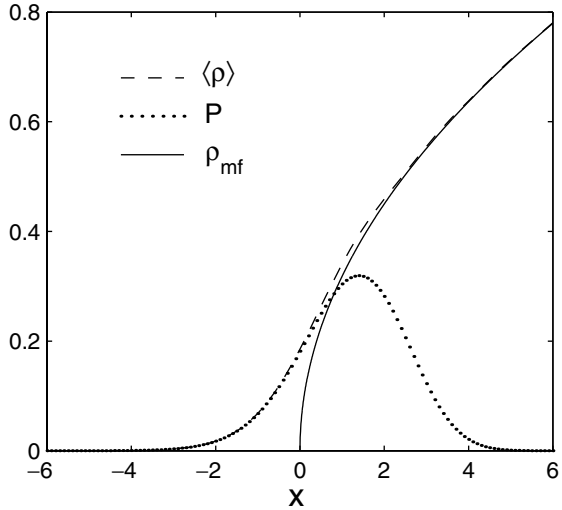

FIG. 2. Mean-field and ensemble averaged density of states ρ_{mf} and $\langle\rho\rangle$, together with the probability distribution P of the excitation gap, as a function of the rescaled energy $x = (\varepsilon_1 - E_g)/\Delta_g$. These curves are the universal predictions of the random-matrix theory.

breaking magnetic field. The starting point of our calculation is the effective Hamiltonian for a quantum dot coupled to a superconductor [10],

$$\mathcal{H} = \begin{pmatrix} H & -\pi W W^\dagger \\ -\pi W W^\dagger & -H^* \end{pmatrix}. \tag{2}$$

Here H is an $M \times M$ Hermitian matrix representing the Hamiltonian of the isolated quantum dot, and W is an $M \times N$ matrix that describes the coupling to the superconductor via an N-mode point contact. For a ballistic point contact, $W_{mn} = \pi^{-1}\delta_{mn}(M\delta)^{1/2}$ [11]. The number M is sent to infinity at the end of the calculation [12]. The effective Hamiltonian is a valid description of the low-lying excitations if the rate $N\delta$ of collisions with the superconductor (i.e., the escape rate from the normal quantum dot) is much smaller than the order parameter Δ of the bulk superconductor. See Ref. [10] for a microscopic derivation of Eq. (2). In the absence of a magnetic field, the matrix H is symmetric. To describe an ensemble of chaotic quantum dots (or disordered metal grains), we take H from the Gaussian orthogonal ensemble (GOE) of random-matrix theory [13],

$$\mathcal{P}(H) \propto \exp\left(-\frac{\pi^2}{4\delta^2 M}\,\mathrm{Tr}H^2\right). \tag{3}$$

The choice of the distribution (3) is justified, since both characteristic energy scales E_g and Δ_g of our problem are small compared to the inverse ergodic time $\hbar/\tau_{\mathrm{erg}}$. (This is the Thouless energy of the isolated quantum dot.) In this case, validity of random-matrix theory for the Hamiltonian H of the isolated quantum dot is known to be valid for dots with diffusive [14] and ballistic chaotic [15] electron dynamics.

Calculation of the density of states of $\mathcal{H}$ using perturbation theory in N^{-1} yields the result (1) discussed in the introduction. Our problem is to go beyond perturbation theory and find the probability distribution $P(\varepsilon_1)$ of the lowest positive eigenvalue ε_1 of the Hamiltonian (2).

We have solved this problem numerically by exact diagonalization of the effective Hamiltonian $\mathcal{H}$. Before presenting these results, we first describe an entirely different approach, which leads to an analytical prediction for the gap distribution. We invoke the universality hypothesis of random-matrix theory, that the local spectral statistics of a chaotic system depends only on the symmetry properties of the Hamiltonian, and not on microscopic properties. This universality hypothesis has been proven for a broad class of Hamiltonians in the bulk of the spectrum [16] but is believed to be valid near the edge of the spectrum as well. A proof exists for so-called trace ensembles, having $\mathcal{P}(H) \propto \exp[-\mathrm{tr}f(H)]$, with f an arbitrary polynomial function [17].

The mean-field density of states near the edge can be written in the form

$$\rho_{\mathrm{mf}}(\varepsilon) = \frac{1}{a}\left(\frac{\varepsilon - b}{a}\right)^p, \qquad \varepsilon > b\,. \tag{4}$$

According to the universality hypothesis, the spectral statistics near the edge, in rescaled variables $(\varepsilon - b)/a$, depends only on the exponent p and on the symmetry index β [$\beta = 1$ (2) in the presence (absence) of time-reversal symmetry]. Generically, p is either $1/2$ (soft edge) or $-1/2$ (hard edge). For our problem, we have $\beta = 1$, $p = 1/2$, $a = \pi^{2/3}\Delta_g$, $b = E_g$; cf. Eq. (1). The corresponding gap distribution is given by [18]

$$P(\varepsilon) = \frac{d}{d\varepsilon} F_1[(\varepsilon - E_g)/\Delta_g], \tag{5}$$

$$F_1(x) = \exp\left(-\tfrac{1}{2}\int_{-\infty}^{x}[q(x') + (x - x')q^2(x')]\,dx'\right). \tag{6}$$

The function $q(x)$ is the solution of

$$q''(x) = -xq(x) + 2q^3(x)\,, \tag{7}$$

with asymptotic behavior $q(x) \to \mathrm{Ai}(-x)$ as $x \to -\infty$ [$\mathrm{Ai}(x)$ being the Airy function].

The distribution (5) is shown in Fig. 3 (solid curve). It is centered at a positive value of $x = (\varepsilon_1 - E_g)/\Delta_g$, meaning that the average gapsize $\langle\varepsilon_1\rangle$ is about Δ_g bigger than the mean-field gap E_g. For small x there is a tail of the form

$$P(x) \approx \frac{1}{4\sqrt{\pi}\,|x|^{1/4}}\,\exp(-\tfrac{2}{3}|x|^{3/2}), \qquad x \ll -1\,. \tag{8}$$

Nonuniversal corrections to the distribution (5) become important for energy differences $|\varepsilon - E_g| \gtrsim E_g$, hence for $|x| \gtrsim N^{2/3}$. Since the width of the gap distribution

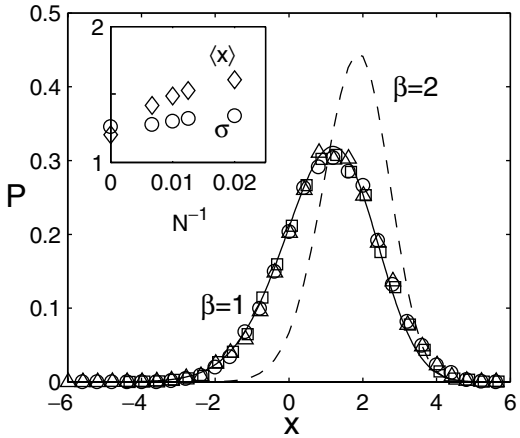

FIG. 3. Probability distribution of the rescaled excitation gap $x = (\varepsilon_1 - E_g)/\Delta_g$. Data points follow from an exact diagonalization of 10^4 realizations of the effective Hamiltonian (2) for different values of M and N ($\triangle$: $M = 400$, $N = 200$; $\square$: $M = 600$, $N = 150$; $\bigcirc$: $M = 600$, $N = 80$). The solid curve is the universal prediction (5) of random-matrix theory. The mean of the data points has been adjusted to fit the curve by applying a horizontal offset; no other fit parameters are involved. The inset shows the actual mean $\langle x \rangle$ and root-mean-square value σ of the data for $M/N = 4$ for different values of N, together with the random-matrix prediction for $N \to \infty$. These results are all in zero magnetic field. The dashed curve is the random-matrix theory prediction (15) in the presence of a time-reversal symmetry breaking magnetic field ($\beta = 2$).

is of order unity in the variable x, the probability to find a sample with an excitation gap in the nonuniversal regime is exponentially small.

In order to verify our universality hypothesis, we compare Eq. (5) with the results of an exact diagonalization of the Hamiltonian (2). As one can see in Fig. 3, the numerical data are in good agreement with the analytical prediction. The small deviations can be attributed to the finiteness of N and M in the numerics.

Let us now consider the effect of a weak magnetic field on the gap distribution. In the effective Hamiltonian, the presence of a magnetic field is described by replacing H by [19]

$$H(\alpha) = H + i\alpha A. \tag{9}$$

Here A is an $M \times M$ real antisymmetric matrix, whose off-diagonal elements have the same variance as those of H. The parameter α is proportional to the magnetic field,

$$M\alpha^2 = \eta\left(\frac{\Phi}{\Phi_0}\right)^2 \frac{\hbar}{\tau_{\rm erg}\delta}, \tag{10}$$

where Φ is the magnetic flux through the quantum dot, $\Phi_0 = h/e$ is the flux quantum, and η is a nonuniversal numerical constant [11]. The case $\alpha = 0$ corresponds to the GOE that we considered above; the case $\alpha = 1$ corresponds to the Gaussian unitary ensemble (GUE) of fully broken time-reversal symmetry.

The effect of a magnetic field on the density of states in mean-field theory is known [4]. The square-root singularity (1) near the gap still holds, but the magnitude of the gap is reduced. The critical flux Φ_c at which $E_g = 0$ and hence the proximity effect is fully suppressed is given by

$$M\alpha^2 \sim N \Rightarrow \Phi_c \sim \Phi_0\sqrt{\frac{N\tau_{\rm erg}\delta}{\hbar}}. \tag{11}$$

This is a much larger flux than the flux $\Phi_{\rm bulk}$ at which the spectral statistics in the bulk of the spectrum crosses over from GOE to GUE, which is given by [19]

$$M\alpha^2 \sim 1 \Rightarrow \Phi_{\rm bulk} \sim \Phi_0\sqrt{\frac{\tau_{\rm erg}\delta}{\hbar}}. \tag{12}$$

We now argue that the characteristic flux $\Phi_{\rm edge}$ for the spectral statistics at the edge of the spectrum is intermediate between Φ_c and $\Phi_{\rm bulk}$. We consider the effect of the magnetic field on the lowest eigenvalue ε_1 of $\mathcal{H}$ to second order in perturbation theory,

$$\delta\varepsilon_1 = \sum_{j\neq 1} \alpha^2 \frac{|\langle 1|\mathcal{A}|j\rangle|^2}{\varepsilon_1 - \varepsilon_j}, \qquad \mathcal{A} = i\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}. \tag{13}$$

Since typically $|\langle 1|\mathcal{A}|2\rangle|^2 \sim M\delta^2/\pi^2$ and $\varepsilon_2 - \varepsilon_1 \sim \Delta_g$, we see that the effect of level repulsion from the neighboring level ε_2 on the lowest level ε_1 becomes comparable to $\Delta_g \sim N^{1/3}\delta$ if

$$M\alpha^2 \sim N^{2/3} \Rightarrow \Phi_{\rm edge} \sim \Phi_0\sqrt{\frac{N^{2/3}\tau_{\rm erg}\delta}{\hbar}}. \tag{14}$$

The terms in Eq. (13) with $j \gg 1$ give a uniform shift of all low-lying levels and, hence, do not affect the fluctuations. For $N \gg 1$ the flux scale (14) for breaking time-reversal symmetry at the edge of the spectrum is much smaller than the critical flux Φ_c needed to suppress the proximity effect. Indeed, using $N \sim E_g/\delta$ we find $\Phi_{\rm edge} \sim \Phi_0\tau_{\rm erg}^{1/2}E_g^{1/3}\delta^{1/6}$, which is much smaller than $\Phi_c \sim \Phi_0\tau_{\rm erg}^{1/2}E_g^{1/2}$. Notice that the naive substitution of δ by Δ_g in expression (12) for $\Phi_{\rm bulk}$ would give the wrong result for $\Phi_{\rm edge}$.

To study numerically the crossover in the gap fluctuation statistics, $N^{2/3} \ll N$ has to be satisfied, which is difficult. The analytical prediction for fully broken time-reversal symmetry is [18]

$$P(\varepsilon) = \frac{d}{d\varepsilon} F_2[(\varepsilon - E_g)/\Delta_g], \tag{15}$$

$$F_2(x) = \exp\left(-\int_{-\infty}^{x} (x - x')q^2(x')\,dx'\right). \tag{16}$$

This curve is shown dashed in Fig. 3. The tail for small x is now given by

$$P(x) \approx \frac{1}{8\pi|x|}\exp(-\tfrac{4}{3}|x|^{3/2}), \qquad x \ll -1. \tag{17}$$

The gap fluctuations are significantly reduced by

TABLE I. Characteristic energy and magnetic flux scales for the spectral statistics in the bulk and at the edge of the spectrum and for the size of the gap.

	Energy scale	Flux scale
Bulk statistics	δ	$\Phi_0 \tau_{\rm erg}^{1/2} \delta^{1/2}/\hbar^{1/2}$
Edge statistics	$E_g^{1/3}\delta^{2/3}$	$\Phi_0 \tau_{\rm erg}^{1/2} \delta^{1/6} E_g^{1/3}/\hbar^{1/2}$
Gap size	E_g	$\Phi_0 \tau_{\rm erg}^{1/2} E_g^{1/2}/\hbar^{1/2}$

application of a magnetic field; see Fig. 3 and Eqs. (8) and (17). The suppression of the fluctuations is a generic feature of the different level statistics for ensembles with orthogonal and unitary symmetries, the ensembles with less symmetry (the unitary ensemble) having a more rigid and, hence, less fluctuating spectrum [13].

To make contact with Ref. [9] we briefly discuss the implications of our results for the ensemble averaged density of states $\langle \rho(\varepsilon) \rangle$ in the subgap regime. The tail of $P(x)$ for $x \lesssim -1$ is the same as the tail of $\langle \rho \rangle$; cf. Fig. 2. We conclude that [20]

$$\langle \rho(x) \rangle \propto \exp\left(-\frac{2\beta}{3} x^{3/2}\right) \tag{18}$$

over a broad range $\Delta_g \ll E_g - \varepsilon \ll E_g$ inside the mean-field gap. A different exponential decay (with a power of 2 instead of 3/2 in the exponent) was predicted recently by Beloborodov, Narozhny, and Aleiner [9], for the subgap density of states of an ensemble of superconducting grains in a weak magnetic field. Since the mean-field density of states in that problem is also of the form (1), the universal GUE edge statistics should apply. The reason that the universal decay (18) was not obtained in Ref. [9] is that their theory applies to the nonuniversal energy range $\varepsilon \ll E_g$ near the Fermi level. To emphasize the significance of the universal energy range we note that the probability to have the lowest energy level in that range is larger than in the nonuniversal range by an exponentially large factor $\propto \exp[(E_g/\Delta_g)^{3/2}]$.

In conclusion, we have argued that the proximity effect in a mesoscopic system has a gap distribution which is universal once energy is measured in units of the energy scale $\Delta_g \propto (E_g \delta^2)^{1/3}$ defined from the mean-field density of states $\rho(\varepsilon) = [(\varepsilon - E_g)/\Delta_g^3]^{1/2}/\pi$. This universal distribution is the same as the distribution of the smallest eigenvalue of the Gaussian orthogonal or unitary ensembles from random-matrix theory, depending on whether time-reversal symmetry is broken or not. We have identified the magnetic field scale for breaking time-reversal symmetry and verified our results by exact diagonalization of an effective Hamiltonian. Characteristic energy and magnetic field scales are summarized in Table I. The universality of our prediction should offer ample opportunities for experimental observation.

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