We analyze the effect of environment on the gate operation of flux-biased phase qubits. We employ the master equation for a reduced density matrix of the qubit system coupled to an Ohmic environment, described by the Caldeira-Leggett model. Numerically solving this equation, we evaluate the gate error as a function of gate time, temperature, and environmental coupling strength for experimentally determined qubit parameters. The analysis is presented for single-quadrature microwave (control) pulses as well as for two-quadrature pulses, which lower the gate error significantly for idealized systems in the absence of environment. Our results indicate that two-quadrature pulses with fixed and variable driving frequency have similar performance, which outweighs the performance of single-quadrature pulses, in the presence of environment.

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I. INTRODUCTION

Superconducting circuits containing Josephson junctions are promising candidates for a scalable quantum-information processing.1–5 However, small separations between successive quantum energy states in these circuits4,5 do not permit selective manipulation of the qubit in a two-dimensional subspace and results in a dynamical leakage of a quantum state to a broader Hilbert space of the circuit.6 To reduce this leakage, Motzoi et al.7 proposed a derivative removal by adiabatic gate (DRAG) method, which reduces the gate error to $10^{-5}$ for an experimentally optimal gate time of 6 ns. This error is well below the required error threshold of $10^{-3}$ for fault tolerant quantum computation.8

In addition to the dynamic leakage, any realistic model of a qubit must also address coupling of the qubit to environment, which leads to further destruction of qubit states. Several efforts have already been made toward the study of accurate control of a qubit system.6,9,10 However, the effect of an environment on optimally controlled qubit has only been studied in a phenomenological model,7 which leads to the evolution of density matrix of the qubit in Lindblad form.11

In this paper, we resort to a microscopic approach to the modeling of the environment. We employ the Caldeira-Leggett model of the system-environment coupling12,13 to describe time evolution of a flux-biased phase qubit, driven by the DRAG pulses.7 Numerically solving equation of motion for the qubit density matrix, we study the dependence of the gate error on temperature, gate time, and environmental coupling strength.

Although numerous potential sources of decoherence in phase qubits have been studied experimentally,14–18 in this paper, we focus on decoherence due to the Gaussian noise, which is introduced to the qubit system within the Caldeira-Leggett model. The study of the effect of ubiquitous low-frequency $1/f$ noise on the gate error is out of the scope of this paper.

We specifically study the role of dissipation in the gate error of NOT gate operation. We find that for phase qubits with relaxation time $T_1 \approx 700$ ns,19 two-quadrature DRAG pulses proposed in Ref. 7 result in the gate error exceeding $7 \times 10^{-3}$, which is too high for fault tolerant quantum computation, for a gate time of 6 ns. We then address the limitation posed by the environmental coupling on two-quadrature pulses. Here we find that for optimal DRAG pulses,7 the coupling to environment must be reduced nearly by a factor of 6 to suppress the gate error below the required threshold. We also investigate the gate error for simple pulse shaping where the pulse amplitude of the first quadrature varies smoothly according to a Gaussian-shaped function while the amplitude of the second quadrature is proportional to the derivative of the first. In this case, however, the microwave drive frequency is fixed. For this pulse shape, we conclude that the gate error reduces to $10^{-5}$ for a gate time $\approx 7$ ns when the coupling to environment is reduced by an approximate factor of 10 from the coupling in currently used phase qubits.

II. MODEL

A flux-biased phase qubit consists of a Josephson junction (JJ) embedded in a superconducting loop.2 Finite resistance of the JJ results in dissipation processes in the qubit and can be accounted for by the Caldeira-Leggett model.12,13 The full Hamiltonian of the qubit and the environment is

$$\hat{H} = \hat{H}_q + \hat{P}(t) + \hat{H}_R + \hat{V}. \quad (1)$$

The Hamiltonian of the qubit $\hat{H}_q$ is written in terms of operators $\hat{Q}$ and $\hat{\phi}$, the charge and phase difference of the JJ, respectively,

$$\hat{H}_q = \frac{\hat{Q}^2}{2C} + \frac{\phi_0}{2\pi} \left[ \frac{\phi_0}{4\pi L} \left( \delta - \frac{2\pi \phi_{\text{ext}}}{\phi_0} \right)^2 - I_0 \cos \hat{\phi} \right]. \quad (2)$$

where $L$ ($C$) is the loop inductance (junction capacitance), $\phi_{\text{ext}}$ is the external magnetic flux applied to the phase qubit, $I_0$ is the critical current of the JJ, and $\phi_0 = \hbar/2e$ is the flux quantum. The qubit is capacitively coupled to microwave current source, used to induce coherent transitions between the qubit states.2 This coupling introduces time-dependent part in the Hamiltonian.
Here $I(t) = I_c(t) \cos \omega_d t + I_c(t) \sin \omega_d t$ is microwave current with frequency $\omega_d$.

The environment is introduced as a set of harmonic oscillators (reservoir) with the Hamiltonian

$$\hat{H}_R^{(N)}(t) = \sum_{n=1}^{N} \left( \hbar \omega_n \hat{a}_n^\dagger \hat{a}_n + \frac{\hbar}{2} \hat{a}_n^2 \right).$$

The coupling between the qubit system and the reservoir is bilinear in the JJ phase $\hat{\delta}$ and oscillator displacements $\hat{\delta}_\alpha$

$$\hat{V} = \sum_{\alpha=1}^{N} \gamma_\alpha \hat{\delta}_\alpha \hat{\delta}, \quad \hat{q} = \hat{\delta} - \frac{2\pi \phi_{\text{ext}}}{\phi_0},$$

where parameters $\gamma_\alpha$ determine the coupling strength between the qubit and reservoir mode $\alpha$.

Our goal is to describe the time evolution of the qubit density matrix $\hat{\rho}(t)$. The qubit is initially prepared in a pure state, corresponding to the density matrix $\hat{\rho}(0)$. Assuming that the environment is in a thermal equilibrium at temperature $T$, the master equation for $\hat{\rho}(t)$ takes the following form:

$$\frac{d\hat{\rho}(t)}{dt} = -i\hbar [\hat{H}_q(t), \hat{\rho}(t)] - \hat{\mathcal{L}}[\hat{\rho}(t)]$$

and the dissipative term is

$$\hat{\mathcal{L}}[\hat{\rho}(t)] = \frac{1}{\hbar^2} \int_0^t dt' \eta_1(t') \{\hat{q}(t), \{\hat{q}(-t'), \hat{\rho}(t)\}\}$$

$$- \frac{1}{\hbar^2} \int_0^t dt' \eta_2(t') \{\hat{q}(t), \{\hat{q}(-t'), \hat{\rho}(t)\}\},$$

where $\hat{q}$ is a Heisenberg operator. In Eq. (5), $\eta_1(t)$ is the damping part and $\eta_2(t)$ represents the quantum noise of the environment.

$$\eta_1(t) = \hbar \int_0^\infty J(\omega) \left[ 1 + 2N(\omega) \right] \cos \omega t \, d\omega,$$

$$\eta_2(t) = \hbar \int_0^\infty J(\omega) \sin \omega t \, d\omega.$$  

The master Eq. (5) is time local, however, it contains time-dependent coefficients, which capture memory effects of the noise due to the heat bath. The spectral density $J(\omega)$ is an emergent quantity of the qubit system. The Planck’s function $N(\omega) = 1/\exp(h\omega/T) - 1$ defines an average excitation number of environment modes with frequency $\omega$.

In typical experiments with phase qubits, the “potential” part of $\hat{H}_q$ in Eq. (2) has one deep minimum and another very shallow minimum that disappears at the critical flux $\phi_c$. External flux $\phi_{\text{ext}}$ is chosen in such a way that only a few levels are localized in the shallow well but these levels are still separated from levels localized in the deep well by impenetrable barrier. As a result, we truncate the qubit Hamiltonian, Eqs. (2) and (3), to three localized levels and obtain the following Hamiltonian:

$$\hat{H}_q(t) = \hbar \sum_{j=1}^{2} \left[ \omega_j \hat{a}_j^\dagger \hat{a}_j + a^* \lambda_j \hat{a}_j^\dagger + \lambda_j \hat{a}_j \right] + \hat{H}_{nr},$$

where $\hat{U}_{ij} = \langle j | \langle i |$ is the projector for the $j$th level, $\hat{a}_j$ is an energy eigenvalue of time-independent Hamiltonian $\hat{H}_q$, and $\hat{H}_{nr}$ contains nonresonant terms. In this three-level model, the lower two energy levels comprise qubit states while the third level accounts for a leakage level.

### III. GATE ERROR AND DRAG METHOD

In order to quantify the error during gate operation we use gate fidelity averaged over two initial input states in a two-dimensional Hilbert space, similar to one defined in Ref. 25

$$F_g = \frac{1}{2} \text{Tr} \left( \hat{U}_{\text{ideal}} \hat{\rho}^{(0)}(t) \hat{U}_{\text{ideal}}^\dagger \hat{\rho}_f(t_f) \right).$$

Here $\hat{U}_{\text{ideal}}$ represents an ideal evolution, $\hat{\rho}_f(0)$ is an actual density matrix of the qubit system with $\hat{\rho}_f(0) = \hat{\rho}_f^{(0)}$, and $\hat{\rho}_f^{(0)}$ represents two initial axial states in a Bloch sphere. The gate error $E$ is defined as $E = 1 - F_g$.

A simple approach to minimize leakage of quantum information from qubit subspace is to use a single-quadrature Gaussian envelope pulse given by

$$I_x(t) = I_x(t) = A e^{-t^2 / (2\sigma^2)} - B, \quad I_y(t) = 0, \quad I_z(t) = 0, \quad \text{for a NOT gate operation},$$

where $t_q$ is a gate time and $\sigma = t_q / 2$. For a NOT gate operation, which we choose to focus on without any loss of generality, constant $B$ is chosen so that the Gaussian pulse starts and finishes off at zero, and $A$ is defined by

$$\int_0^{t_q} I_x(t) dt = \pi.$$  

This pulse shape results in a large gate error for reasonably short pulses.

The DRAG method reduces the gate error to order of $10^{-5}$ for a gate time of 6 ns (Ref. 7) by using two quadratures and time-dependent detuning $d(t) = \omega_0 - \omega_{J}(t) = (\lambda^2 - 4)\eta_2(t)/\Delta$, where the anharmonicity parameter $\Delta = \omega_{J} - 2\omega_0$, and $\lambda$ measures relative strength of $0 \rightarrow 1$ and $1 \rightarrow 2$ transitions, that is, $\lambda = \lambda_{J}/\lambda_{0}$. We note that the laboratory frame is more suitable for the solution of the reduced density matrix of the qubit coupled to the environment. Therefore, we preserve the form of the quadrature amplitudes as in Ref. 7.
\[ I_\pi = I_{\pi} + \frac{(\lambda^2 - 4)I_z^3}{8\Delta^2}, \quad I_y = -\frac{I_{\pi}}{\Delta} \]  

and obtain the following equation for the microwave driving frequency in the Hamiltonian Eq. (10) in the laboratory frame:

\[ i\hbar \omega_0(t) + \omega_d(t) = \omega_0 - d_1(t), \quad \omega_d(0) = \omega_0. \]  

Although the DRAG correction is successful in reducing the gate error below the required threshold, a practical implementation may not be feasible due to stringent requirement to vary microwave frequency. For this reason, we also consider two-quadrature pulses with fixed driving frequency \( \omega_d = \omega_0 \). We transform Hamiltonian (10) to a frame rotating with frequency \( \omega_d \) with respect to the laboratory frame and obtain

\[ \hat{H}^R = \hbar \sum_{j=1}^2 \left[ d_j \hat{I}_j + \frac{I_j(t)}{2} \lambda_j \hat{\sigma}_{j-1,j} + \frac{I_{j-1}(t)}{2} \lambda_j \hat{\sigma}_{j-1,j}^2 \right], \]  

where \( d_j = \Delta + 2d_1 \), and for \( \omega_d = \omega_0 \), the detuning \( d_1 = 0 \). We introduce operators \( \hat{\sigma}_{j,x} = |k\rangle\langle j| + |j\rangle\langle k| \) and \( \hat{\sigma}_{j,x}^2 = |k\rangle\langle j| - |j\rangle\langle k| \).

To analyze the dynamics of rotating frame Hamiltonian \( \hat{H}^R \), it is convenient to perform an adiabatic transformation \( \hat{D}(t) = \exp[-i\hat{I}_1(t) (\alpha \hat{\sigma}_{1,2}^0 + \lambda \hat{\sigma}_{1,2}^0)/2\Delta] \), which preserves the form of the gate, if \( I_1(t) \) starts and finishes off at zero. This condition is satisfied by our choice of \( I_1(t) \) [see Eq. (12a)]. The parameter \( \alpha \) appearing in \( \hat{D} \) is a dimensionless scaling parameter. After performing the transformation, the Hamiltonian, to first order in \( I_1/\Delta \), takes the following form:

\[ \frac{\hat{H}^D}{\hbar} = I_\pi \frac{\lambda^2}{4\Delta} \hat{\Pi}_1 + \left[ \frac{\alpha I_\pi^2}{4\Delta} \hat{\Pi}_1 + \frac{I_\pi}{2\Delta} \lambda \hat{\sigma}_{1,2} + \frac{2 - \alpha}{8\Delta} \hat{\sigma}_{0,2}^2 + \frac{\lambda(\alpha - 1)I_\pi}{4\Delta} \hat{\sigma}_{0,2}^2 \right]. \]  

We then require resonant condition for the microwave \( \pi \) pulse in the qubit subspace and also eliminate the imaginary inertial term from the subspace, that is, require \( \hat{\Pi}_1 \) and \( \hat{\sigma}_{0,1}^0 \) terms in Eq. (15) to vanish, to obtain

\[ \alpha = \frac{\lambda^2}{4}, \quad I_\pi(t) = I_\pi(t), \quad I_{\pi}(t) = -\frac{\alpha I_\pi(t)}{\Delta}, \]  

where \( I_{\pi}(t) \) is defined by Eq. (12a). The contributions to the gate error due to transitions to the third level come from the second and third lines of Eq. (15) except for \( \hat{\Pi}_1 \) term, which is not directly responsible for the gate error. Using the above expression for \( I_{\pi}(t) \) and Eq. (12b), we estimate the magnitude of these terms as

- \( \hat{\sigma}_{1,2}^0 : \left[ \frac{I_\pi}{2\Delta} + \frac{I_{\pi}}{2} \right] \sim \frac{1}{\Delta \tau_g^2} \),
- \( \hat{\sigma}_{0,2}^0 : \left( 2 - \alpha \right) \frac{\lambda I_{\pi}^2}{8\Delta} \sim \frac{1}{\Delta \tau_g^2} \),
- \( \hat{\sigma}_{0,2}^0 : \frac{\lambda(\alpha - 1)I_\pi}{4\Delta} \sim \frac{1}{\Delta \tau_g^2} \).

These estimates show that the error due to \( \hat{\sigma}_{1,2}^0 \) and \( \hat{\sigma}_{0,2}^0 \) terms are comparable and results in the leading contribution to the gate error. In the case of time-dependent detuning, the choice of pulses is such that it eliminates the error associated with \( \hat{\sigma}_{1,2}^0 \) term and rescaling of the pulse intensity \( I_{\pi}^2 \) term in Eq. (13a) removes the contribution to the gate error due to \( \hat{\sigma}_{0,2}^0 \) term. This elimination of \( 1/\Delta \tau_g^2 \) explains high effectiveness of variable driving frequency DRAG pulses. For fixed frequency pulses, the pulse rescaling only marginally reduces the gate error.

\[ \text{FIG. 1. (Color online) Gate error vs gate time in log-normal scale with (thick lines) and without (thin lines) dissipation for a single quadrature Gaussian (\( \sigma=0.5\tau_g \)) pulse (solid black), the Gaussian (\( \sigma=0.5\tau_g \)) pulse with first-order DRAG correction and dynamical detuning (dashed-dotted blue), and the Gaussian (\( \sigma=0.5\tau_g \)) pulse with fixed driving frequency \( \omega_d=\omega_0 \) and \( \alpha=0.5 \) (dashed red), all in the laboratory frame. For the dissipative case, temperature \( T=0.1\hbar \omega_0 \), the cut-off frequency \( \omega_0=10\omega_0 \), and the coupling parameter \( \xi=2 \).} \]

\[ \hat{\sigma}_{1,2}^0 : \left[ \frac{I_\pi}{2\Delta} + \frac{I_{\pi}}{2} \right] \sim \frac{1}{\Delta \tau_g^2}, \]

\[ \hat{\sigma}_{0,2}^0 : \left( 2 - \alpha \right) \frac{\lambda I_{\pi}^2}{8\Delta} \sim \frac{1}{\Delta \tau_g^2}, \]

\[ \hat{\sigma}_{0,2}^0 : \frac{\lambda(\alpha - 1)I_\pi}{4\Delta} \sim \frac{1}{\Delta \tau_g^2}. \]

\[ \text{IV. RESULTS} \]

Qubit parameters used below in our simulation are typical of phase qubits: \( C=1 \) pF, \( I_0=1.5 \mu \text{A}, \beta_I = 2\pi I_0 L / \phi_0 = 3.2 \), and \( \phi_m = 0.955 \phi_c \), where \( \phi_c \) is a critical flux. Numerical simulation indicates that small variations of qubit parameters do not incur any noticeable change in the gate error as long as there are at least three energy levels in the shallow well of the potential. For these experimental parameters, we numerically solve the time-independent Schrödinger’s equation with the Hamiltonian given by Eq. (2). From this simulation, we obtain the following numerical values (rounded up to two decimal places): \( \omega_0 = 39.43 \) GHz, \( \lambda = 1.42 \), and \( \Delta = -2.43 \) GHz.

In Fig. 1, we plot the gate error for the DRAG pulses with and without time-dependent detuning for an ideal phase qu-
bit without environment. We find that pulses with two quadratures and fixed driving frequency (thin dashed red) perform much better than single quadrature Gaussian pulses (thin solid black) but are not as effective as pulses with double quadratures and time-dependent driving frequency (thin dashed-dotted blue).

We verify numerically that the fixed frequency DRAG pulses give the minimal gate error for the choice of parameter \( \alpha \) according to Eq. (16). As shown in Fig. 2(a), minimum value of the error occurs at around \( \alpha=0.5 \) for different gate times, cf. dashed blue curve for \( t_g \omega_0=250 \) and solid black curve for \( t_g \omega_0=350 \). This result is consistent with Eq. (16), since for the phase qubit \( \lambda=1.42 \), which implies \( \alpha=0.5 \). For transmon qubits, discussed in Ref. 27, \( \alpha=0.4 \) owing to different value of \( \lambda \).

In order to study the effect of dissipation on the DRAG pulses, we integrate the master Eq. (5) numerically using the fourth- and fifth-order Runge-Kutta method. First, we consider relaxation of the qubit from the first excited state to the ground state in the absence of microwave drive, shown in Fig. 2(b). For this simulation, we choose the cut-off frequency \( \omega_0=10\omega_0 \) (throughout this paper), temperature \( T=0.1\hbar\omega_0 \), and the coupling parameter \( \xi=2 \) so that the relaxation time corresponds to experimentally observed decay time \( T_1 \approx 700 \) ns for phase qubits.\(^{19}\) We note that spontaneous relaxation rate of the first excited state can also be evaluated from the master Eq. (5) as

\[
\Gamma = \frac{1}{T_1} = 2\pi\hbar\omega_0^2 \frac{\xi C}{4c^2} |q_{01}|^2, \quad q_{01} = (0|\hat{q}|1). \tag{17}
\]

For the above choice of the dimensionless coupling parameter \( \xi \), temperature \( T \), and the cut-off frequency \( \omega_0 \), we study the effect of dissipation on two-quadrature pulses.

In Fig. 1, we observe a nonmonotonic behavior of the gate error with gate time for pulses with the DRAG corrections. We find that for shorter gate times, two-quadrature pulses with time-dependent driving frequency are less affected by dissipation (thick dashed-dotted blue). However, for longer gate times, dissipation has a substantial effect on two-quadrature pulses. For instance, for a gate time of \( \omega_{g} = 250 \) (\( t_g = 6 \) ns), the gate error increases from \( 10^{-5} \) to higher order of \( 10^{-3} \) for two-quadrature pulses with dynamical detuning, when dissipation is taken into account. This increase in the gate error is due to relaxation of the qubit from excited state to the ground state, which becomes prominent for longer gate times. For comparison, we plot the gate error for three different types of pulses: single-quadrature Gaussian pulse (thick solid black), the Gaussian pulse with first-order DRAG correction and time-dependent driving frequency (thick dashed-dotted blue) and the Gaussian pulse with two quadratures and fixed driving frequency (thick dashed red). One can conclude from these plots that the performance of two-quadrature pulses without detuning is comparable to the DRAG pulses with dynamical detuning when dissipation is included.

Next, we study the effect of the environmental coupling strength on the gate error. In Fig. 3, we plot the gate error for different coupling parameters \( \xi \) for the phase qubit driven by two-quadrature pulses with dynamical detuning. In this simulation, we consider temperature \( T=0.1\hbar\omega_0 \), and coupling parameters: \( \xi=0 \) (thin dashed-dotted blue), \( \xi=0.1 \) (thick solid green with circles), \( \xi=0.3 \) (thick solid red), and \( \xi=2 \) (thick dashed-dotted blue). At \( \xi=0 \) the gate error originates entirely due to microwave-induced leakage of the qubit state from the lowest two level subspace. The environment-induced transition rates increase with increase in the environmental coupling strength \( \xi \) [see Eq. (17)]. As a result, the gate error also increases, which is corroborated by Fig. 3. One can infer from the plot that two-quadrature pulses with time-dependent driving frequency suppress the gate error to \( 10^{-5} \) for \( \xi=0.3 \) and a gate time of \( \omega_{g} t_g = 200 \) (\( t_g = 5.5 \) ns). This indicates that an increase in the relaxation time nearly by a factor of 6 from the currently observed value is necessary to suppress the error below the threshold.

We further analyze the effect of the environmental coupling on fixed frequency two-quadrature pulses for a range of gate times. For this case, gate errors for different values of \( \xi \) are plotted in Fig. 4, where temperature is the same as above and coupling parameters are: \( \xi=0 \) (thin dashed red), \( \xi=0.1 \) (thick solid green with circles), \( \xi=0.2 \) (thick solid blue), \( \xi=0.3 \) (thick solid red), and \( \xi=2 \) (thick dashed-dotted blue).
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FIG. 4. (Color online) Gate error vs gate time in log-normal scale for the Gaussian ($\sigma=0.5t_g$) pulse with fixed frequency DRAG correction for temperature $T=0.1h\omega_0$ and cut-off frequency $\omega_c = 10\omega_0$. The environmental coupling parameters: $\xi=0$ (thin dashed red), $\xi=0.1$ (thick solid green with circles), $\xi=0.2$ (thick solid blue), $\xi=0.5$ (thick solid black with triangles), and $\xi=2$ (thick dashed red).

=0.5 (thick solid black with triangles), and $\xi=2$ (thick dashed red). These plots indicate that the DRAG pulses with fixed driving frequency can effectively suppress the gate error if the environmental coupling strength is weakened and gate times are slightly longer than 6 ns. More specifically, for $\xi=0.2$ and a gate time of $\omega_0 t_g = 300$ ($t_g$ = 7 ns), the gate error is close to $10^{-3}$. Therefore, we conclude that the relaxation time must be nearly a factor of 10 longer than the currently observed value to attain the threshold of the gate error for fixed frequency DRAG pulses. This is a much better improvement compared to single-quadrature pulses for which the gate error never reduces to the threshold for a reasonable choice of gate times even in an ideal case, that is, $\xi=0$, as shown in Fig. 1.

Finally, we investigate the effect of temperature on the gate error. In Fig. 5, we plot the gate error normalized to the error at zero temperature for two different gate times: $\omega_0 t_g = 150$ (dashed-dotted blue) and $\omega_0 t_g = 350$ (dashed black). The plot shows a monotonic growth of the gate error as temperature increases due to enhancement in the relaxation rate.

We compare results of numerical solution of the master Eq. (5) and the simple picture of the error due to coupling to the environment in terms of the “Fermi-Golden rule” transition rates. Considering the environment at zero temperature and assuming that the contribution to the error $E$ from the environment is small, $E \ll 1$, we can evaluate the error as the probability of an excitation of a reservoir mode during the qubit operation, which happens with rate $E(T=0) = \Gamma_g\rho_{11}(t)$, where $\rho_{11}(t) = \int_0^t \rho_{11}(t')dt'$. $\rho_{11}(t)$ is the time average of probability of qubit being in the first excited state. At finite temperature, processes with excitation of environment happen with rate $\Gamma(T)=\Gamma[1+N(\omega_0)]$. In addition, the qubit can absorb an excitation from the environment with rate $\Gamma N(\omega_0)$. We combine the qubit excitations from the ground to first state and the first to second state with the relaxation from the first to ground state, and obtain the following estimate for the gate error due to coupling to the environment:

$$E(T) \approx 1 + 4N(\omega_0).$$

For an average occupation of the ground and first excited states being $\approx 1/2$, and for a weak anharmonicity of the qubit system $|\Delta| \ll \omega_0$, the gate error reduces to

$$E(T) \approx 1 + 4N(\omega_0).$$

The estimated normalized gate error (solid red) is plotted in Fig. 5 together with the gate error obtained from the numerical simulation. The rate equation estimation of the error is fairly close to the error obtained from direct numerical simulation for a longer gate time (dashed black). However, for a shorter gate time (dashed-dotted blue), the estimated error deviates from the exact numerical simulation considerably suggesting that the rate equation description may not be valid for shorter gate times and higher temperatures.

V. DISCUSSION AND CONCLUSIONS

In this paper we compared possible choices of microwave pulses for NOT gate operation in fluxed-biased qubits. Particularly, we considered three options: single-quadrature pulses and two-quadrature microwave (control) pulses with both variable and fixed frequency. Two-quadrature pulses led to significant suppression of the gate error compared to single-quadrature pulses. However, the presence of dissipative environment increased the gate error even for two-quadrature pulses significantly above the required threshold for fault tolerant quantum computation in currently available phase qubits. We further investigated how the environmental coupling strength affects the gate error and found that an improvement of the qubit relaxation time is crucial for effectiveness of the DRAG pulses. We determined that two-quadrature pulses with fixed driving frequency suppress the gate error below the required threshold for a reasonable gate
time of 7 ns but for qubits with the relaxation time ten times longer than the currently observed relaxation times. Similarly, our analysis indicated that two-quadrature pulses with dynamical detuning can also effectively reduce the gate error below the required threshold if the relaxation time is longer by at least a factor of 6. In comparison to fixed frequency DRAG pulses, this is a moderate improvement over the longer relaxation-time requirement, yet not impressive enough to outshine the difficulty associated with implementing control pulses with variable driving frequency. We expect that in a trade-off between complicated driving frequency and longer relaxation times, the DRAG pulses with fixed frequency are viable alternatives for reducing the gate error. We emphasize that for single-quadrature pulses, reduction in the gate error below the error threshold of 10^{-3} is not possible for reasonable gate times, even in an ideal case without any dissipation.

In addition, we observed a monotonic increase in the gate error with temperature, which is due to increase in the relaxation rate with temperature. We found that temperature dependence of the gate error for longer pulses can be captured by a simple error estimation based on the rate equations. Nonetheless, the simple estimation of the error for shorter pulses differs from the gate error obtained from direct numerical solution of the reduced density matrix. Therefore, full density-matrix solution is necessary to calculate the error for shorter gate times.

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24. The levels in the deep well can also be accounted in the present model, however, our numerical results indicate that the gate error does not change significantly if those levels are also included in the calculation for chosen values of parameters.
26. We also made constant detuning of the driving frequency from \omega_0 but did not see any improvement compared to \omega_0 case.