Statistical Mechanics, Physics 715
Homework Assignment 1, due October 5, 2009

Problem 1. Prove the following thermodynamic relations:

a) $$\left(\frac{\partial C_P}{\partial P}\right)_T = -T\left(\frac{\partial^2 V}{\partial T^2}\right)_P;$$

b) $$\left(\frac{\partial E}{\partial P}\right)_T = -T\left(\frac{\partial V}{\partial T}\right)_P - P\left(\frac{\partial V}{\partial P}\right)_T;$$

c) $$\left(\frac{\partial E}{\partial T}\right)_P = C_P - P\left(\frac{\partial V}{\partial T}\right)_P;$$

d) $$\left(\frac{\partial T}{\partial V}\right)_S = -\frac{T}{C_V}\left(\frac{\partial S}{\partial V}\right)_T;$$

e) $$\left(\frac{\partial T}{\partial P}\right)_S = -\frac{T}{C_P}\left(\frac{\partial S}{\partial P}\right)_T;$$

f) $$\left(\frac{\partial V}{\partial T}\right)_P = -\frac{C_P}{T}\left(\frac{\partial T}{\partial V}\right)_S\left(\frac{\partial V}{\partial P}\right)_S;$$

g) $$\left(\frac{\partial P}{\partial V}\right)_S = \left(\frac{\partial P}{\partial V}\right)_T - \frac{T}{C_V}\left(\frac{\partial P}{\partial T}\right)_V^2.$$

Problem 2. Consider a Carnot engine that operates between a heater with heat capacity $C_{\rm h}$ and a reservoir with heat capacity $C_{\rm r}$. The initial temperatures of the heater and reservoir are $T_{\rm h}^{(0)}$ and $T_{\rm r}^{(0)}$, respectively; $T_{\rm h}^{(0)} > T_{\rm r}^{(0)}$. Assume that the heater releases its internal energy only to the engine and the reservoir absorbs energy only from the engine and that in each cycle the change in temperatures of the heater and reservoir are infinitesimally small.
a) Calculate the work, that could be performed by such engine.
b) Find the limiting expression for the work for $C_{\rm r} \to \infty$. Explain, why this work is smaller than the change in internal energy of the heater.

Problem 3. Consider a process in which work done in a cycle of a Carnot engine is entirely applied to operate a refrigerator. The Carnot engine is operating between temperatures $T_1 > T_2$ and the refrigerator is operating between temperatures $T_3 < T_2$. Calculate the amount of heat Q released in this process at temperature T_2, if the amount absorbed from the heater by the Carnot engine is equal to $Q_{\rm h}$. Find the efficiency $\eta = Q/Q_{\rm h}$ of this process. Explain why $\eta > 1$ does not contradict to the laws of thermodynamics.

Problem 4. Huang, Problem 1-4.

Problem 5. Consider a thermodynamic system containing $N \gg 1$ of non-interacting two level systems with energy states $\pm\Delta/2$. The energy states of the system are $E_m = \Delta m/2$, where $|m| \le N$.
a) What is the degeneracy Γ_m of these quantum levels?
b) Calculate the partition function $Z(T) = \sum_{|m|\le N} \Gamma_m \exp(-E_m/T)$ and the free energy $F(T) = -T\ln Z(T)$. (Hint: for $N \gg 1$ replace the sum over m by an integral and use the saddle point approximation).
c) Calculate entropy $S(T) = -\partial F(T)/\partial T$ and heat capacitance of the system as a function of temperature T.
d) For microcanonical ensemble with energy E we can introduce temperature T as a solution of the equation $E = F(T) + TS(T)$. Find $T = T(E)$ for $|E| \le \Delta N/2$.
e) If the system is in contact with another system at temperature $T_1 > 0$, what is the sign of energy change for these two systems?