Lecture 5

Momentum operator

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}.$$

Average value (Expectation value) is given by

$$\langle p_x\rangle = \int_{-\infty}^{+\infty} \psi^*(x)\,\hat{p}_x\,\psi(x)\,dx.$$

p is a real number;

$p = p^*$; Indeed,

$$p^* = \int_{-\infty}^{+\infty}\left(\psi^*(x)\left(-i\hbar\partial_x\right)\psi(x)\right)^* dx$$

$$= i\hbar\int_{-\infty}^{+\infty}\psi(x)\frac{\partial}{\partial x}\psi^*(x)\,dx \overset{\text{Int by parts}}{=} -i\hbar\int \psi^*(x)\frac{\partial}{\partial x}\psi(x)\,dx$$

$$= p.$$

What could be an appropriate state with momentum p?

$\hat{p}_x\psi_p(x) = p\,\psi_p(x)$, which is often called an eigen state of the momentum operator $\hat{p}_x$.

$$p = \int \psi_p^*(x)\,\hat{p}_x\,\psi_p(x)\,dx = p\int|\psi_p(x)|^2 dx = p.$$

$\langle p_x^2\rangle = p^2$; $\langle p_x^2\rangle - \langle p_x\rangle^2 = 0$, its variance is $\underline{\underline{\text{ZERO}}}$.

Let us look for a solution of

$$\hat{p}_x\psi_p(x) = p\psi_p(x); \quad -i\hbar\frac{\partial}{\partial x}\psi_p(x) = p\psi_p(x)$$

$$\psi_p(x) = C e^{\frac{ipx}{\hbar}}$$

However $\int \psi_p^*(x)\psi_p(x)\,dx = C^2\int dx$ — diverges.

A conventional "normalization" condition is $\int \psi_p^*(x)\psi_{p'}(x)\,dx = \text{const}\ \delta(p-p')$.

$\boxed{3.4}$ Commutation relation for operators of momentum and coordinate

$$(\hat{p}_x \cdot \hat{x} - \hat{x}\,\hat{p}_x)\psi(x)$$

$$= -i\hbar\left[\partial_x(x\psi(x)) - x\partial_x\psi(x)\right] =$$

$$= -i\hbar\hat{1}\psi(x),$$ we use the notation

$$[\hat{p}_x; \hat{x}] = -i\hbar.$$

This is the second example of non-commuting variables. $[\hat{S}_x; \hat{S}_z] \neq 0.$

If the Hamiltonian has a potential term, $U(x)$, then $[\hat{p}, \hat{H}] \neq 0$,

$$p\,U(x) - U(x)\,p = -i\hbar\,\partial_x U(x) \leftarrow \text{'force'}.$$

Consider a wave-function $\psi(x)$, such that

$\int x|\psi(x)|^2 dx = 0$ and $\int \psi^* p_x \psi \, dx = 0$

Let us calculate $\langle x^2 \rangle = \delta x^2 = \int x^2 |\psi(x)|^2 dx$

and $\int \psi^*(x)(\hat{p}_x)^2 \psi(x)\, dx = \langle p_x^2 \rangle = \delta p^2$

The inequality $$\int_{-\infty}^{+\infty} \left|\alpha x \psi(x) + \frac{d\psi}{dx}\right|^2 dx \geq 0$$ for any α.

5.3

$$\alpha^2 \int x^2 |\psi(x)|^2 dx = \alpha^2 \delta x^2$$

$$\int x\left(\frac{d\psi^*}{dx}\psi + \psi^*\frac{d\psi}{dx}\right)dx = \int x \frac{d|\psi(x)|^2}{dx} dx$$

$$= - \int |\psi(x)|^2 dx = -1$$ (normalization condition)

$$\int \frac{d\psi^*}{dx}\frac{d\psi}{dx} dx = - \int \psi^* \frac{d^2\psi}{dx^2} dx = \frac{1}{\hbar^2}\int \psi^* p_x^2 \psi \, dx$$

$$= \delta p^2/\hbar^2$$

We obtain $\alpha^2 \delta x^2 - \alpha + \frac{\delta p_x^2}{\hbar^2} \geq 0$

This inequality is satisfied if

$$1 - \frac{4\delta x^2 \delta p_x^2}{\hbar^2} \geq 0, \qquad \delta x \cdot \delta p \geq \frac{\hbar}{2}$$

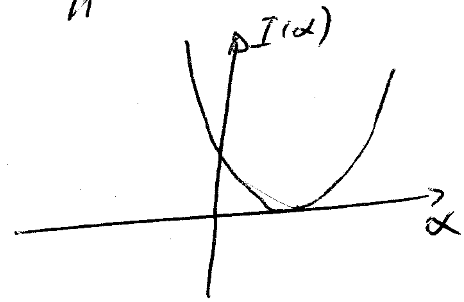
This is the exact form of the Heisenberg (1925) uncertainty relation for momentum and coordinate.

$$\psi(x) = \frac{1}{\sqrt[4]{2\pi}\sqrt{\delta x}} \cdot e^{\frac{ip_0 x}{\hbar}} e^{-x^2/4\delta x^2}.$$

5.4 Let us investigate if $\delta x \cdot \delta p = \frac{\hbar}{2}$ for some configuration of wave-functions.

$$\delta x^2 = \frac{\hbar^2}{4\delta p^2}; \qquad \alpha^2 \frac{\hbar^2}{4\delta p^2} - \alpha + \frac{\delta p^2}{\hbar^2} = 0$$

$$\alpha_{1,2} = \frac{2\delta p^2}{\hbar^2}$$



$I(\alpha) = 0$ is only possible if

$$\alpha x \psi(x) + \frac{d\psi}{dx} = 0 \Rightarrow \ln\psi = -\frac{2\delta p^2}{\hbar^2}\frac{x^2}{2} + C_0$$

$C_1 = e^{C_0}$

$$\psi = C_1 e^{-\frac{\delta p^2 x^2}{\hbar^2}} = C_1 e^{-x^2/4\delta x^2}$$

This wave function is normalizable;

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1 \text{ gives } C_1 = \frac{1}{(2\pi)^{1/4}\sqrt{\delta x}}$$

$$\psi(x) = \frac{1}{(2\pi)^{1/4}\sqrt{\delta x}} e^{-x^2/4\delta x^2}$$

Useful integrals $I_{2n} = \int_{-\infty}^{+\infty} x^{2n} e^{-x^2/a^2} dx = 2\sqrt{\pi}\,\frac{2n!}{n!}\left(\frac{a}{2}\right)^{2n+1}$

$$I_0 = 2\sqrt{\pi}\,\frac{a}{2} = \sqrt{\pi}\, a$$

$$I_2 = 2\sqrt{\pi}\cdot\frac{2!}{1!}\,\frac{a^3}{8} = \frac{\sqrt{\pi}}{2}a^3.$$

$\boxed{5.5}$

$$\langle \delta x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \frac{e^{-x^2/2\delta x^2}}{\sqrt{2\pi}\,\delta x}\,dx = \frac{1}{\sqrt{2\pi}\,\delta x}\cdot 2\sqrt{2}\,\delta x^3 \cdot \frac{\sqrt{\pi}}{2}$$

$$= \delta x^2 \text{ (As expected)}$$

$\langle p_x \rangle = 0$ (symmetry)

$$\langle p_x^2 \rangle = -\hbar^2 \int_{-\infty}^{+\infty} \psi^*(x) \frac{d^2}{dx^2} \psi(x)\,dx$$

$$= +\hbar^2 \int_{-\infty}^{+\infty} \left|\frac{d\psi}{dx}\right|^2 dx = \frac{\hbar^2}{\sqrt{2\pi}\,\delta x} \cdot \int_{-\infty}^{+\infty} \left|\frac{x}{2\delta x^2}\right|^2 e^{-x^2/2\delta x^2}\,dx$$

$$= \frac{\hbar^2}{\sqrt{2\pi}\,\delta x^5} \cdot \frac{1}{4} \int_{-\infty}^{+\infty} x^2 e^{-x^2/2\delta x^2}\,dx =$$

$$= \frac{1}{4}\frac{\hbar^2}{\sqrt{2\pi}\,\delta x^5} \cdot 2\sqrt{2}\,\delta x^3 \cdot \frac{\sqrt{\pi}}{2} = \frac{\hbar^2}{4\delta x^2} = \delta p^2$$

This wave packet is called as the minimal wave packet, since the uncertainty in δp and δx is minimized.

<u>Teaser question</u>

$$\psi(x) = \begin{cases} 0, & x < -\frac{a}{2} \\ \sqrt{a^2 - x^2}, & -\frac{a}{2} < x < \frac{a}{2} \\ 0, & x > \frac{a}{2} \end{cases} \qquad \psi(x) = \frac{1}{\sqrt{a}} \cos\frac{\pi x}{2a}$$

$\sqrt{\frac{3}{4}}\frac{1}{a^{3/2}}$