<u>Lecture 22</u>

$$\psi_l^m(\theta,\varphi) = Y_l^m(\theta,\varphi) = \dots$$

These functions are orthogonal,

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\varphi \left(Y_l^m(\theta,\varphi)\right)^* Y_{l'}^{m'}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}.$$

for $m \neq m'$, the above is straitforward:

$$\int_0^{2\pi} d\varphi \; e^{-im\varphi} e^{+im'\varphi} = \delta_{mm'} \cdot 2\pi$$

For equal m and m'; but different l & l' the proof is more complicated, but we know, that eigenfunctions of a Hermitian operator, corresponding to different eigenvalues, are orthogonal.

The Hamiltonian of a 3d rotator:

$$\hat{H} = \frac{\hat{L}^2}{2J},$$ and J is the moment of inertia.

In quantum mechanics, the allowed values of energy are

$$E_l = \frac{\hbar^2}{2J} l(l+1)$$ and eigenfunctions coincide with eigenfunction of the square of the angular momentum operator $\hat{L}^2$

Spin

States with half-integer angular momentum are not allowed, because the rotation around on 2π changes sign.

For elementary particles such rotations have no meaning and states with half-integer ~~angular~~ values of the angular momentum operators are allowed. These states are intrinsic characteristics of elementary particles.

To distinguish from the real angular momentum we use the notation for operators

$$\hat{S} = \{\hat{S}_x ; \hat{S}_y ; \hat{S}_z\}$$

$$[\hat{S}_x ; \hat{S}_y] = i\hbar \hat{S}_z ; \quad [\hat{S}_y ; \hat{S}_z] = i\hbar \hat{S}_x \quad [\hat{S}_z ; \hat{S}_x] = i\hbar \hat{S}_y$$

It is convenient to redefine $\hat{S} \to \frac{\hat{S}}{\hbar}$

$$[\hat{S}_x ; \hat{S}_y] = i \hat{S}_z$$

22.3

For most elementary particles, $s = 1/2$

where s is the "eigen value" of $\hat{S}^2$:

$$\hat{S}^2 |s, m\rangle = s(s+1) |s; m\rangle \qquad S_z |s; m\rangle = m |s; m\rangle$$

$s = 1/2$ $\qquad \frac{3}{4}$ $\qquad m = \pm\frac{1}{2}$.

Any state can be represented as a linear superposition of $|1/2; \pm 1/2\rangle$:

$$|\chi\rangle = \alpha_+ |1/2; +1/2\rangle + \alpha_- |1/2; -1/2\rangle;$$

$$|\alpha_+|^2 + |\alpha_-|^2 = 1.$$

Such representation requires only two complex numbers $(\alpha_+; \alpha_-)$

$$|\chi\rangle \longleftrightarrow \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$$

$$|1/2; +1/2\rangle \leftrightarrow \begin{pmatrix} \alpha_+ = 1 \\ \alpha_- = 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1/2; -1/2\rangle \leftrightarrow \begin{pmatrix} \alpha_+ = 0 \\ \alpha_- = 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

21.4) The corresponding representation of the spin operators is realized in terms of 2×2 matrices:

$$\hat{\vec{S}}^2 |\chi_+\rangle = \frac{3}{4} |\chi_+\rangle \quad \text{or} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 0 \end{pmatrix}$$

$$a_{11} = 3/4; \quad a_{21} = 0$$

$$\hat{\vec{S}}^2 |\chi_-\rangle = \frac{3}{4} |\chi_-\rangle \quad \begin{pmatrix} 3/4 & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/4 \end{pmatrix}$$

$$a_{12} = 0, \quad a_{22} = 3/4$$

$$\hat{\vec{S}}^2 = \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix}$$

Similarly: $\hat{S}_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\hat{S}_+ |\chi_-\rangle = |\chi_+\rangle; \quad \hat{S}_+ |\chi_+\rangle = 0$$

gives $\hat{S}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Similarly, $\hat{S}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

21.5)

$$\hat{S}_x = \frac{S_+ + S_-}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{from} \quad \hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-)$$

A convenient (common) notation is

$$\hat{\vec{S}} = \frac{1}{2}\{\hat{\sigma}_x; \hat{\sigma}_y; \hat{\sigma}_z\} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{1}{4} \cdot 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{\sigma}_i \hat{\sigma}_k = 1\,\delta_{ik} + i\sum_l \epsilon_{ikl}\hat{\sigma}_l$$

$$\epsilon_{ikl} = \begin{cases} 1 & \text{if} \quad ikl = 123;\ 312;\ 231 \\ -1 & \text{if} \quad ikl = 321\ \ 132\ \ 213 \\ 0 & \text{otherwise} \end{cases}$$

$$\sigma_x \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_z.$$