The following problem in electrostatics is traditionally considered difficult by the students and the teachers. Here we present its full solution.

**Jackson 3.3:** A thin, flat, conducting, circular disc of radius $R$ is located in the $x-y$ plane with its center at the origin, and is maintained at a fixed potential $V$. With the information that the charge density on a disc at fixed potential is proportional to $(R^2 - r^2)^{-1/2}$, where $r$ is the distance out from the center of the disc,

a) show that for $r > R$ the potential is

$$\Phi(r, \theta, \phi) = \frac{2V}{\pi} \frac{R}{r} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left( \frac{R}{r} \right)^{2n} P_{2n}(\cos \theta);$$

b) find the potential for $r < R$;

c) what is the capacitance of the disc?

**Solution:** We write the Green function (assuming that the potential is zero at $r \to \infty$) in spherical coordinates as:

$$G(x, x') = -\frac{1}{4\pi |x - x'|} = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_l^l}{r_{>l}^l} Y_{lm}(\theta, \phi)Y_{lm}(\theta', \phi'),$$

(1)

where $r_<$ is the smaller of the two parameters $r = |x|$ and $r' = |x'|$, and $r_>$ is the larger. The charge density concentrated on the disc is represented in spherical coordinates by

$$\rho(x)d^3x = \frac{Q}{2\pi R} \theta(R - r) \frac{\delta(\cos \theta)}{r \sqrt{R^2 - r^2}} d\cos \theta d\phi r^2 dr,$$

(2)

where $\theta(R - r)$ is a Heaviside step function, and $Q$ is the charge of the disc. One can easily verify that

$$\int \rho(x)d^3x = Q.$$  

(3)

With the aid of the Green function the potential is found in the whole space as:

$$\Phi(x) = -\frac{1}{\epsilon_0} \int G(x, x') \rho(x')d^3x'.$$

(4)

Since only the $m = 0$ terms survive, we get:

$$\Phi(r, \theta) = \frac{Q}{4\pi \epsilon_0 R} \sum_{l=0}^{\infty} I_l(r) P_l(0) P_l(\cos \theta),$$

(5)

and since only the even-order Legendre polynomials are nonzero at $\cos \theta = 0$, we get:

$$\Phi(r, \theta) = \frac{Q}{4\pi \epsilon_0 R} \sum_{n=0}^{\infty} I_{2n}(r) P_{2n}(0) P_{2n}(\cos \theta).$$

(6)
In these expressions the integral $I_l(r)$ has the form for $r < R$:

\[
I_l(r) = \int_0^R \frac{r'\,dr'}{\sqrt{R^2 - r'^2}} \equiv \int_0^{r'_{l+1}} \frac{r'^{l+1}\,dr'}{\sqrt{R^2 - r'^2}} + \int_r^{r_{l+1}} \frac{r'^l\,dr'}{r\sqrt{R^2 - r'^2}},
\]

and for $r > R$:

\[
I_l(r) = \int_0^R \frac{r'\,dr'}{\sqrt{R^2 - r'^2}} \equiv \int_0^{r_{l+1}} \frac{r'^{l+1}\,dr'}{\sqrt{R^2 - r'^2}}.
\]

These integrals can be especially easily calculated for the limit of vanishing $r$:

\[
I_{2n}(r)|_{r \to 0} = \int_0^R \frac{dr'}{\sqrt{R^2 - r'^2}} = \frac{\pi}{2} \delta_{n,0},
\]

which means that the only nonvanishing integral corresponds to $n = 0$. The other simple case is $r > R$:

\[
I_{2n}(r)|_{r \geq R} = \int_0^R \frac{r'^{2n+1}\,dr'}{r^{2n+1}\sqrt{R^2 - r'^2}} = \left( \frac{R}{r} \right)^{2n+1} \frac{1}{2} \int_0^1 \frac{t^n\,dt}{\sqrt{1-t}} = \left( \frac{R}{r} \right)^{2n+1} \frac{(2n)!!}{(2n+1)!!},
\]

where the integral is done by effecting the change of variables $t = (r'/R)^2$, and by integrating by parts $n$ times. The limiting case (9) allows us to evaluate the potential at the center of the disc,

\[
\Phi_0 = \frac{\pi}{2} \frac{Q}{4\pi \epsilon_0 R},
\]

and since $\Phi = V$ at the disc we can express the potential everywhere in space as:

\[
\Phi(r,\theta) = \frac{2V}{\pi} \sum_{n=0}^\infty I_{2n}(r) P_{2n}(0) P_{2n}(\cos \theta).
\]

Formula (12) provides the general solution of the problem, where the functions $I_{2n}(r)$ are given by (7, 8).

We now discuss in more detail the solution for $r > R$ and $r < R$. The first case is relatively simple. The second case often causes confusion, and it will be discussed in more detail.

**The region $r \geq R$.** Substituting in (12) the expression (10), and noting that

\[
P_{2n}(0) = \frac{(-1)^n(2n-1)!!}{(2n)!!},
\]

we derive the outside potential:

\[
\Phi_{\text{out}}(r,\theta) = \frac{2V}{\pi} \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} \left( \frac{R}{r} \right)^{2n+1} P_{2n}(\cos \theta),
\]

which coincides with Jackson’s solution.

**The region $r < R$.** In this case the functions $I_{2n}(r)$ are not powers of $r$, and the structure of the solution is not that transparent. We can however expand these functions in the powers of $r$, by using the following expansion in $y = r/R < 1$:

\[
\frac{1}{\sqrt{1-y^2}} = \sum_{m=0}^\infty \frac{(2m-1)!!}{(2m)!!} y^{2m}
\]

\[
2
\]
inside the integrals in (7) and by integrating each term:

\[
I_{2n}(r) = \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \frac{y^{2m+1}}{(2m+2n+2)} - \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \frac{y^{2m+1}}{(2m-2n+1)} + \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \frac{y^{2n}}{(2m-2n+1)}.
\] (16)

Here the first term comes from the first integral in (7), and the last two terms from the second integral. The last sum in (16) can be done in a closed form:

\[
\sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \frac{y^{2n}}{(2m-2n+1)} = \frac{\pi}{2} \delta_{n,0},
\] (17)

meaning that this sum is identically zero, except for the case \(n = 0\). This result is not immediately obvious; it can be most easily derived by representing the second integral in (7) through the integral in a complex \(z\)-plane over a closed contour encircling the cut made along the interval \((0, R)\). One then notices that the residue at \(z = \infty\) is nonzero only if \(n = 0\). We leave this exercise to the reader. Combining the remaining two sums together we simplify:

\[
I_{2n}(r) = \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \frac{(-1)(4n+1)}{(2m+2n+2)(2m-2n+1)} \left(\frac{r}{R}\right)^{2m+1} + \frac{\pi}{2} \delta_{n,0}.
\] (18)

We will now consider the obtained solution separately in the half intervals \(0 \leq \cos \theta \leq 1\) and \(-1 \leq \cos \theta < 0\). In the whole interval \([-1, 1]\) the Legendre polynomials form a complete and mutually orthogonal system of functions in \(L^2\) space, so the expansion of our solution in Legendre polynomials (12) is unique. In each of the half intervals, however, the system of even Legendre polynomials and the system of odd Legendre polynomials independently provide complete sets of mutually orthogonal functions. A function defined in a given half interval can be expanded there in either odd or even Legendre polynomials. Our result (12) provides the expansion of the solution in the even Legendre polynomials. We will demonstrate now that in each of the half intervals the expansion of the solution in the odd Legendre polynomials has a much simpler form.

For that we need to know how to expand the odd Legendre polynomials in the even Legendre polynomials in the interval \([0, 1]\). The answer is [see e.g., Byerly, 1959, An Elementary Treatise on Fourier’s Series and Spherical, Cylindrical, and Ellipsoidal Harmonics, page 173]:

\[
\int_{0}^{1} P_{2m+1}(\mu) P_{2n}(\mu) d\mu = \frac{(-1)^{m+n}(2n-1)!!(2m+1)!!}{(2n)!!(2m)!!(2n+2m+2)(2m-2n+1)}.
\] (19)

Recalling that the Legendre polynomials are normalized according to:

\[
\int_{0}^{1} P_{l}(\mu) P_{l}(\mu) d\mu = \frac{1}{2l+1},
\] (20)

we derive the required expansion:

\[
P_{2m+1}(\cos \theta) = \sum_{n=0}^{\infty} C_{m,n} P_{2n}(\cos \theta),
\] (21)

where

\[
C_{m,n} = \frac{(-1)^{m+n}(2n-1)!!(2m+1)!!(4n+1)}{(2n)!!(2m)!!(2n+2m+2)(2m-2n+1)}.
\] (22)
Combining (12), (18), (21), and (22), we see that in the region $0 \leq \cos \theta \leq 1$ and $r < R$ (that is, above the disc) our solution (12) can be rewritten in a very simple form:

$$\Phi(r, \theta) = V + \frac{2V}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{2m+1} \left( \frac{r}{R} \right)^{2m+1} P_{2m+1}(\cos \theta),$$  \hspace{1cm} (23)

and, analogously, in the region $-1 \leq \cos \theta \leq 0$ and $r < R$ (below the disc) we get (inverting the sign of the odd Legendre polynomials):

$$\Phi(r, \theta) = V - \frac{2V}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{2m+1} \left( \frac{r}{R} \right)^{2m+1} P_{2m+1}(\cos \theta).$$  \hspace{1cm} (24)

Note that the potential is equal to $V$ at the disc ($\cos \theta = 0$), and the derivative of the potential normal to the disc is discontinuous across the disc, which, as the reader may check, is consistent with the discontinuity of the normal component of the electric field due to the disc charge density.

**The answer:** Formula (12) or its simplifications in different regions (14), (23), and (24) constitute the complete solution of the problem. We summarize it here:

$$\Phi(r, \theta) = \frac{2V}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left( \frac{R}{r} \right)^{2n+1} P_{2n}(\cos \theta), \quad r \geq R,$$  \hspace{1cm} (25)

$$\Phi(r, \theta) = V + \frac{2V}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{2m+1} \left( \frac{r}{R} \right)^{2m+1} P_{2m+1}(\cos \theta), \quad r \leq R, \quad 0 \leq \theta \leq \pi/2,$$  \hspace{1cm} (26)

$$\Phi(r, \theta) = V - \frac{2V}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{2m+1} \left( \frac{r}{R} \right)^{2m+1} P_{2m+1}(\cos \theta), \quad r \leq R, \quad \pi/2 \leq \theta \leq \pi.$$  \hspace{1cm} (27)

It may seem that the above solution expanded in the odd Legendre polynomials for $r < R$ cannot match at $r = R$ the solution expanded in the even Legendre polynomials for $r \geq R$, which sometimes leads to the (incorrect) conclusion that the answer given in Jackson is wrong. The confusion stems from the fact that one does not recognize that for $r < R$, the solution is not expanded in the odd Legendre polynomials $P_{2m+1}(\cos \theta)$ in the whole interval $[-1, 1]$. Rather, it is expanded in this interval in the functions $P_{2m+1}(\cos \theta) \text{sgn}(\cos \theta)$, which are not orthogonal to the even Legendre polynomials. As unobvious as it may seem, the inner solution (26)-(27) does match the outer solution (25) at $r = R$.

The two different series (12) and (26) converge pointwise to the same function in the interval $[0, 1]$. The derivatives of these series with respect to $\cos \theta$, however, converge to the same function only in $(0, 1)$, but not in $[0, 1]$. At the point $\cos \theta = 0$, the derivative of (12) has a discontinuity, $\Phi_{(12)}'(\cos \theta \to 0) \neq \Phi_{(12)}'(0) = 0$, while the derivative of (26) is continuous $\Phi_{(26)}'(\cos \theta \to 0) = \Phi_{(26)}'(0) \neq 0$. The Laplacian of (12) is also discontinuous, $\nabla^2 \Phi_{(12)}(\cos \theta \neq 0) = 0$, $\nabla^2 \Phi_{(12)}(0) = \infty$, while the Laplacian of (26) is continuous, $\nabla^2 \Phi_{(26)}(\cos \theta \neq 0) = \nabla^2 \Phi_{(26)}(0) = 0$.

One therefore needs to use the even Legendre polynomials for expanding the Green function, as the source $\delta(\cos \theta)$ is expandable in the even Legendre polynomials while it is not expandable in the odd ones. Indeed, the delta function does not belong to $L^2$, and it should not necessarily be expandable in a set of functions complete in $L^2[0,1]$. Alternatively, when one solves the problem without using the Green function, but rather by expanding the solutions in orthogonal functions and matching them at the boundary, it is convenient to use the odd Legendre polynomials for $r < R$, as each of them automatically satisfies the boundary condition $P_{2m+1} = \text{const} = 0$ at the disc.

**The capacitance of the disc.** According to formula (11), the disc capacitance is $C = Q/V = 8\epsilon_0 R$. 

4