Jackson 3.3

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The (correct!) answer is given in Jackson. The problem is, however, traditionally considered difficult by students and teachers. Here we present its full solution.

Jackson 3.3: A thin, flat, conducting, circular disc of radius R is located in the $x-y$ plane with its center at the origin, and is maintained at a fixed potential V. With the information that the charge density on a disc at fixed potential is proportional to $(R^2-\rho^2)^{-1/2}$, where ρ is the distance out from the center of the disc,
a) show that for $r>R$ the potential is

$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{R}{r}\right)^{2n} P_{2n}(\cos\theta);$$

b) find the potential for $r<R$;
c) what is the capacitance of the disc?

Solution:
The answer is given by formulae (26, 27, 28) below. There are several ways to solve this problem. One way is to expand the potential and the charge density in the Legendre polynomials and to solve the resulting equations for the coefficients of expansion. Here we use a different method, based on the Green function.

We write the Green function (assuming that the potential is zero at $r\to\infty$) in spherical coordinates as:

$$G(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi), \tag{1}$$

where $r_<$ is the smaller of the two parameters $r=|\mathbf{x}|$ and $r'=|\mathbf{x}'|$, and $r_>$ is the larger. The charge density concentrated on the disc is represented in spherical coordinates by

$$\rho(\mathbf{x})d^3x = \frac{Q}{2\pi R}\delta(\cos\theta)\frac{\theta(R-r)}{r\sqrt{R^2-r^2}}\, d\cos\theta\, d\phi\, r^2 dr, \tag{2}$$

where $\theta(R-r)$ is a Heaviside step function, and Q is the charge of the disc. One can easily verify that

$$\int \rho(\mathbf{x})d^3x = Q. \tag{3}$$

With the aid of the Green function the potential is found in the whole space as:

$$\Phi(\mathbf{x}) = -\frac{1}{\epsilon_0}\int G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')d^3x'. \tag{4}$$

Since only the $m=0$ terms survive, we get:

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0 R}\sum_{l=0}^{\infty} I_l(r)P_l(0)P_l(\cos\theta), \tag{5}$$

and since only the even-order Legendre polynomials are nonzero at $\cos\theta = 0$, we get:

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0 R}\sum_{n=0}^{\infty} I_{2n}(r)P_{2n}(0)P_{2n}(\cos\theta). \tag{6}$$

In these expressions the integral $I_l(r)$ has the form for $r < R$:

$$I_l(r) = \int_0^R \frac{r_<^l}{r_>^{l+1}}\frac{r'dr'}{\sqrt{R^2 - r'^2}} \equiv \int_0^r \frac{r'^{l+1}dr'}{r^{l+1}\sqrt{R^2 - r'^2}} + \int_r^R \frac{r^l dr'}{r'^l\sqrt{R^2 - r'^2}}, \tag{7}$$

and for $r > R$:

$$I_l(r) = \int_0^R \frac{r_<^l}{r_>^{l+1}}\frac{r'dr'}{\sqrt{R^2 - r'^2}} \equiv \int_0^R \frac{r'^{l+1}dr'}{r^{l+1}\sqrt{R^2 - r'^2}}. \tag{8}$$

These integrals can be especially easily calculated for the limit of vanishing r:

$$I_{2n}(r)|_{r\to 0} = \int_0^R \frac{dr'}{\sqrt{R^2 - r'^2}} = \frac{\pi}{2}\delta_{n,0}, \tag{9}$$

which means that the only nonvanishing integral corresponds to $n = 0$. The other simple case is $r > R$:

$$I_{2n}(r)|_{r\geq R} = \int_0^R \frac{r'^{2n+1}dr'}{r^{2n+1}\sqrt{R^2 - r'^2}} = \left(\frac{R}{r}\right)^{2n+1}\frac{1}{2}\int_0^1 \frac{t^n dt}{\sqrt{1-t}} = \left(\frac{R}{r}\right)^{2n+1}\frac{(2n)!!}{(2n+1)!!}, \tag{10}$$

where the integral is done by effecting the change of variables $t = (r'/R)^2$, and integrating by parts n times. The limiting case (9) allows us to evaluate the potential at the center of the disc,

$$\Phi_0 = \frac{\pi}{2}\frac{Q}{4\pi\epsilon_0 R}, \tag{11}$$

and since $\Phi = V$ at the disc we can express the potential everywhere in space as:

$$\boxed{\Phi(r,\theta) = \frac{2V}{\pi}\sum_{n=0}^{\infty} I_{2n}(r)P_{2n}(0)P_{2n}(\cos\theta).} \tag{12}$$

Formula (12) provides the general solution of the problem, where the functions $I_{2n}(r)$ are given by (7, 8). We now discuss in more detail the solution for $r > R$ and $r < R$. The first case is relatively simple. The second case often causes confusion, and it will be discussed in more detail.

The region $r > R$. Substituting in (12) expression (10), and noting that[1]

$$P_{2n}(0) = \frac{(-1)^n(2n-1)!!}{(2n)!!}, \tag{13}$$

we derive the outside potential:

$$\Phi_{out}(r,\theta) = \frac{2V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{R}{r}\right)^{2n+1} P_{2n}(\cos\theta), \tag{14}$$

[1]Here and in what follows we assume that when $n = 0$, both $(2n)!!$ and $(2n-1)!!$ are replaced by 1.

which coincides with Jackson's solution.

The region $r < R$. In this case the functions $I_{2n}(r)$ are not powers of r, and the structure of the solution is not that transparent. We can however expand these functions in the powers of r, by using the following expansion in $y = r/R < 1$:

$$\frac{1}{\sqrt{1-y^2}} = \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} y^{2m}. \tag{15}$$

Substituting this expansion inside the integrals in (7) and integrating each term we get:

$$I_{2n}(r) = \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \frac{y^{2m+1}}{(2m+2n+2)} - \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \frac{y^{2m+1}}{(2m-2n+1)} + \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \frac{y^{2n}}{(2m-2n+1)}. \tag{16}$$

Here the first term comes from the first integral in (7), and the last two terms from the second integral. The last sum in (16) can be done in a closed form. For that we notice that this sum can be represented through the integral in a complex z-plane as follows:

$$\sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \frac{y^{2n}}{(2m-2n+1)} = \frac{y^{2n}}{4} \oint_C \frac{dz}{z^n \sqrt{z}\sqrt{1-z}}, \tag{17}$$

where the integration contour encircles the cut made along the interval $(0, 1)$ as shown in Fig. (1). We leave it to the reader to verify this fact.

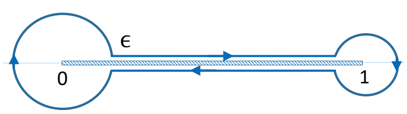

Figure 1: Contour of integration C in Eq. (17). Here ϵ is a small parameter, and the limit $\epsilon \to 0$ should be taken in the answer.

By noticing that the residue at $z = \infty$ is nonzero only if $n = 0$, we obtain:

$$\frac{y^{2n}}{4} \oint_C \frac{dz}{z^n \sqrt{z}\sqrt{1-z}} = \frac{\pi}{2} \delta_{n,0}, \tag{18}$$

meaning that the sum in (17) is identically zero except for the case of $n = 0$. Combining the remaining two sums, we then simplify:

$$I_{2n}(r) = \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \frac{(-1)(4n+1)}{(2m+2n+2)(2m-2n+1)} \left(\frac{r}{R}\right)^{2m+1} + \frac{\pi}{2} \delta_{n,0}. \tag{19}$$

We now consider the obtained solution separately in the half intervals $0 \le \cos\theta \le 1$ and $-1 \le \cos\theta \le 0$. In the whole interval $[-1, 1]$ the Legendre polynomials form a complete and mutually orthogonal system of functions in L^2 space, so the expansion of our solution in Legendre polynomials (12) is unique. In each of the half intervals, however, the system of even Legendre polynomials and the system of odd Legendre polynomials independently provide complete sets of mutually orthogonal functions. A function defined in a given half interval can be expanded there in either odd or even Legendre polynomials. Our result (12) provides the expansion of the solution in the even Legendre polynomials. We will demonstrate now that in each of the half intervals the expansion of the solution in the odd Legendre polynomials has a much simpler form.

For that we need to know how to expand the odd Legendre polynomials in the even Legendre polynomials in the interval $[0, 1]$. The answer is [see e.g., Byerly, 1959, *An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics*, page 173]:

$$\int_0^1 P_{2m+1}(\mu)P_{2n}(\mu)d\mu = \frac{(-1)^{m+n}(2n-1)!!(2m+1)!!}{(2n)!!(2m)!!(2n+2m+2)(2m-2n+1)}. \tag{20}$$

Recalling that the Legendre polynomials are normalized according to:

$$\int_0^1 P_l(\mu)P_l(\mu)d\mu = \frac{1}{2l+1}, \tag{21}$$

we derive the required expansion:

$$P_{2m+1}(\cos\theta) = \sum_{n=0}^{\infty} C_{m,n}P_{2n}(\cos\theta), \tag{22}$$

where

$$C_{m,n} = \frac{(-1)^{m+n}(2n-1)!!(2m+1)!!(4n+1)}{(2n)!!(2m)!!(2n+2m+2)(2m-2n+1)}. \tag{23}$$

Combining (12), (19), (22), and (23), we see that in the region $0 \le \cos\theta \le 1$ and $r < R$ (that is, above the disc) our solution (12) can be rewritten in a very simple form:

$$\Phi(r,\theta) = V + \frac{2V}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{2m+1}\left(\frac{r}{R}\right)^{2m+1}P_{2m+1}(\cos\theta), \tag{24}$$

and, analogously, in the region $-1 \le \cos\theta \le 0$ and $r < R$ (below the disc) we get (inverting the sign of the odd Legendre polynomials):

$$\Phi(r,\theta) = V - \frac{2V}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{2m+1}\left(\frac{r}{R}\right)^{2m+1}P_{2m+1}(\cos\theta). \tag{25}$$

Note that the potential is equal to V at the disc ($\cos\theta = 0$), and the derivative of the potential normal to the disc is discontinuous across the disc, which, as the reader may check, is consistent with the discontinuity of the normal component of the electric field due to the disc charge density.

***The answer*:** Formula (12) or its simplifications in different regions (14), (24), and (25) constitute the complete solution of the problem. We summarize it here:

$$\boxed{\Phi(r,\theta) = \frac{2V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{R}{r}\right)^{2n+1}P_{2n}(\cos\theta), \qquad r > R, \quad 0 \le \theta \le \pi,} \tag{26}$$

$$\boxed{\Phi(r,\theta) = V + \frac{2V}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{2m+1}\left(\frac{r}{R}\right)^{2m+1}P_{2m+1}(\cos\theta), \qquad r < R, \quad 0 \le \theta \le \pi/2,} \tag{27}$$

$$\boxed{\Phi(r,\theta) = V - \frac{2V}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{2m+1}\left(\frac{r}{R}\right)^{2m+1}P_{2m+1}(\cos\theta), \qquad r < R, \quad \pi/2 \le \theta \le \pi.} \tag{28}$$

It may seem that the inner solution (27, 28) expanded in the odd Legendre polynomials cannot match the outer solution (26) expanded in the even Legendre polynomials at $r = R$, which sometimes leads to the (incorrect) conclusion that the answer given in Jackson is wrong. The confusion stems from the fact that for $r < R$, the solution is *not* expanded in the odd Legendre polynomials in the whole interval $[-1, 1]$. Rather, it is expanded in this interval in the functions $P_{2m+1}(\cos\theta)\mathrm{sign}(\cos\theta)$, which are *not* orthogonal to the even Legendre polynomials. As unobvious as it may seem, the inner solution (27)-(28) does match the outer solution (26) at $r = R$; this can be verified by using Eqs. (22) and (23).

It is important to note that it would be incorrect to look for the solution in the inner regions $r < R$ (both above and below the disc) in the form similar to Eq. (26),

$$\Phi(r, \theta) = \sum_n C_{2n} \left(\frac{r}{R}\right)^{2n} P_{2n}(\cos\theta). \tag{29}$$

Indeed, such an expansion would hold for a function satisfying the Laplace equation, while the potential in the inner regions does *not* satisfy the Laplace equation. Rather, it satisfies the Poisson equation with the charge density (2) that expansion in the even Legendre polynomials is nonzero in these regions. Similarly, it would be incorrect to look for the solution in the outer region ($r > R$, $0 < \theta < \pi$) in the form

$$\Phi(r, \theta) = \sum_n C_{2n+1} \left(\frac{R}{r}\right)^{2n+2} P_{2n+1}(\cos\theta)\mathrm{sign}(\cos\theta), \tag{30}$$

since the potential must satisfy the Laplace equation in the outer region, while these expansion functions, obviously, do not.

One needs to use the even Legendre polynomials for expanding the Green function, as the source $\delta(\cos\theta)$ is expandable in the even Legendre polynomials while it is not expandable in the odd ones. Indeed, the delta function does not belong to L^2, and it should not necessarily be expandable in a set of functions complete in $L^2(0, 1)$. Alternatively, when one solves the problem without using the Green function, but rather by expanding the solutions in orthogonal functions and matching them at the boundary, it is convenient to use the odd Legendre polynomials for $r < R$, as each of them automatically satisfies the boundary condition $P_{2m+1} = \mathrm{const} = 0$ at the disc.

The capacitance of the disc. According to formula (11), the disc capacitance is

$$C = Q/V = 8\epsilon_0 R. \tag{31}$$