Quantum Mechanics, Physics 531
Final Exam, due May 11, 2009 at 12:05 pm

Problem 1. *(30 points)* Calculate the correction to the spectrum of a harmonic oscillator due to the anharmonic terms to the second order in ϵ:

$$\begin{aligned}\hat{H} &= \hat{H}_0 + \hat{V};\\ \hat{H}_0 &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2; \quad \hat{V} = \frac{\hbar\omega_0}{2}\left(\epsilon\frac{\hat{x}^3}{\lambda_q^3} + \epsilon^2\frac{\hat{x}^4}{\lambda_q^4}\right).\end{aligned}$$

Here $\lambda_q = \sqrt{\hbar/m\omega_0}$.

Problem 2. *(20 points)* Using the semiclassical Borh-Sommerfeld's quantization rule, calculate the energy difference between two adjacent levels of a weakly anharmonic oscillator, described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2 + \epsilon^2\frac{\hbar\omega_0}{2}\frac{\hat{x}^4}{\lambda_q^4},$$

where $\lambda_q = \sqrt{\hbar/m\omega_0}$ and $\epsilon \ll 1$. The following integral is helpful

$$\int_{-1}^{+1} \frac{d\xi}{\sqrt{1-(\xi^2+a^2\xi^4)/(1+a^2)}} = 2K\left(\frac{-a^2}{1+a^2}\right) \approx \pi - \frac{\pi}{4}a^2,$$

where $K(x)$ is an elliptic function and the last approximation is valid for $a \ll 1$.

Compare your answer with the result of the first order perturbation theory.

Problem 3. *(50 points)* Consider the Jaynes-Cummings Hamiltonian, representing the interaction between a harmonic oscillator and a spin 1/2 particle:

$$\hat{H} = \hbar\omega(\hat{a}^+\hat{a}^- + 1/2) + (\hbar\Omega/2)\hat{\sigma}_z + \lambda\left(\hat{a}^+\hat{\sigma}_- + \hat{a}^-\hat{\sigma}_+\right), \tag{1}$$

where the 2×2 matrices $\hat{\sigma}_\pm = (\hat{\sigma}_x \pm i\hat{\sigma}_y)/2$, and $\hat{\sigma}_{x,y}$ are the Pauli matrices, the operators $\hat{a}^\pm$ are the raising and lowering operators of the harmonic oscillator.

a) What are the eigenvalues and eigenstates of this Hamiltonian in the absence of interaction between the oscillator and the spin, i.e. $\lambda = 0$?

b) Evaluate the expectation value of the full Hamiltonian with respect to the non-interacting eigenstates of this system.

c) Calculate the first order perturbation correction to the eigenstates and eigenenergies of the Hamiltonian with the interaction.

d) Calculate exact eigenenergies of the full Hamiltonian.

e) Consider the initial state of the system with spin up and the first excited state of the oscillator. Write down the time evolution of the quantum state and evaluate the expectation value of the spin projection along z direction.