Lecture 43

Jaynes - Cummings Hamiltonian

$$\hat{H} = \underbrace{\hbar\omega_r(\hat{a}^+\hat{a}^- + \tfrac{1}{2})}_{\text{a free harmonic oscillator (EM field)}} + \underbrace{\frac{\hbar\Omega}{2}\hat{\sigma}_z}_{\text{2 level system (spin)}} + \underbrace{\hbar g(a^+\sigma^- + a^-\sigma^+)}_{\text{coupling}}$$

States $|\uparrow\rangle \otimes |n\rangle$ & $|\downarrow\rangle \otimes |n\rangle$

$$\hat{H}|\uparrow, n\rangle = \left[\hbar\omega_r(n+\tfrac{1}{2}) + \frac{\hbar\Omega}{2}\right]|\uparrow; n\rangle + \sqrt{n+1}\,\hbar g\,|\downarrow, n+1\rangle$$

$$\hat{H}|\downarrow; n+1\rangle = \left[\hbar\omega_r(n+\tfrac{3}{2}) + \frac{\hbar\Omega}{2}(-1)\right]|\downarrow, n+1\rangle + \sqrt{n+1}\,\hbar g\,|\uparrow, n\rangle$$

Only states $|\uparrow, n\rangle$ & $|\downarrow; n+1\rangle$ are mixed

$$\begin{pmatrix} \hbar\omega_r(n+\frac{1}{2}) + \frac{\Omega}{2} & g\sqrt{n+1} \\ g\sqrt{n+1} & \omega_r(n+\frac{3}{2}) - \frac{\Omega}{2} \end{pmatrix} \begin{pmatrix} |\uparrow; n\rangle \\ |\downarrow; n+1\rangle \end{pmatrix}$$

$$= \hbar\left[\hat{1}\,\omega_r(n+1) + \begin{pmatrix} \Delta/2 & g\sqrt{n+1} \\ g\sqrt{n+1} & -\Delta/2 \end{pmatrix}\right]\begin{pmatrix} |\uparrow, n\rangle \\ |\downarrow, n+1\rangle \end{pmatrix}$$

43.2

$|\alpha\rangle$ states; $\hat{H}_{2\times 2} |\alpha\rangle = E_\alpha |\alpha\rangle$

$|\beta\rangle$ $\hat{H}_{2\times 2} |\beta\rangle = E_\beta |\beta\rangle$

Characteristic equation

$$\det \begin{pmatrix} \frac{\Delta}{2} - \lambda & g\sqrt{n+1} \\ g\sqrt{n+1} & -\frac{\Delta}{2} - \lambda \end{pmatrix} = -\left(\frac{\Delta^2}{4} - \lambda^2\right) - g^2(n+1) = 0$$

$$\lambda = \sqrt{g^2(n+1) + \frac{\Delta^2}{4}}$$

$\Delta^2 \gg g^2(n+1)$

$$\lambda = \frac{\Delta}{2} + \frac{g^2(n+1)}{\Delta}$$

$$E = \hbar\omega_r(n+1) \pm \hbar\lambda$$

$$E_n = \frac{-\hbar\omega_r + \hbar\Omega}{2} \pm \frac{g^2(n+1)}{\Delta} + \hbar\omega_r(n+1)$$

$$= \hbar\omega_r(n+\tfrac{1}{2}) + \frac{\hbar\Omega}{2} + \frac{g^2}{\Delta}(n+\tfrac{1}{2}) + \frac{g^2}{2\Delta}$$

$$\hbar\omega_r(n+\tfrac{3}{2}) - \frac{\hbar\Omega}{2} - \frac{g^2}{\Delta}(n+\tfrac{3}{2}) - \frac{g^2}{2\Delta}$$

$$|\alpha\rangle = \cos\theta_n |\uparrow, n\rangle + \sin\theta_n |\downarrow, n+1\rangle$$

$$|\beta\rangle = -\sin\theta_n |\uparrow, n\rangle + \cos\theta_n |\downarrow, n+1\rangle$$

$$\cos\theta_n \left(\omega_n(n+\tfrac{1}{2}) + \frac{\Omega}{2}\right)|\uparrow, n\rangle + \cos\theta_n\left(g\sqrt{n+1}\right)|\downarrow, n+1\rangle$$

$$+ \sin\theta_n \left(\omega_r(n+\tfrac{3}{2}) - \frac{\Omega}{2}\right)|\downarrow n+1\rangle + \sin\theta_n\left(g\sqrt{n+1}\right)|\uparrow, n\rangle$$

$$= \omega_r(n+1)\cos\theta\, |\uparrow n\rangle + \lambda\cos\theta_n |\uparrow n\rangle$$

$$\omega_r(n+1)\sin\theta\, |\downarrow n+1\rangle + \lambda\sin\theta_n |\downarrow n+1\rangle$$

Stark + Lamb shift

13.3

$$\cos\theta_n\left(\frac{\Delta}{2}-\lambda\right)+\sin\theta_n\left(g\sqrt{n+1}\right)=0$$

$$\frac{\sin\theta}{\cos\theta}=\tan\theta_n=\cancel{\frac{g\sqrt{n+1}}{\frac{\Delta}{2}-\lambda}}\quad \cancel{g\sqrt{n}}\ \frac{-\frac{\Delta}{2}+\lambda}{g\sqrt{n+1}}$$

$$\cos\theta_n\ g\sqrt{n+1}+\left(\frac{\Delta}{2}+\lambda\right)\sin\theta=0$$

$$\tan\theta=\frac{g\sqrt{n+1}}{\frac{\Delta}{2}+\lambda}\qquad \tan\theta_1\stackrel{?}{=}\tan\theta_2$$

$$\left(\lambda-\frac{\Delta}{2}\right)\left(\lambda+\frac{\Delta}{2}\right)=g^2(n+1)$$

$$\lambda^2-\frac{\Delta^2}{4}=g^2(n+1)\quad \text{satisfied!}$$

$$\Delta=0\qquad \tan\theta=1$$

resonance

$$|\alpha(t)\rangle=e^{-iE_\alpha t/\hbar}|\alpha(0)\rangle$$
$$|\beta(t)\rangle=e^{-iE_\beta t/\hbar}|\beta(0)\rangle$$

Start in $|\uparrow,n\rangle$

$$|\uparrow,n\rangle=\cos\theta|\alpha\rangle+\sin\theta|\beta\rangle$$

$$|\uparrow,n(t)\rangle=\cos\theta|\alpha(t)\rangle-\sin\theta|\beta(t)\rangle$$

$$S_z=\frac{\hbar}{2}\langle *|\hat{\sigma}_z|*\rangle=$$

$$=\frac{\hbar}{2}\left(\langle\alpha(t)|\cos\theta\, e^{\frac{-iE_\alpha t}{\hbar}}+\langle\beta(t)|\sin\theta\, e^{iE_\beta t/\hbar}\right)\hat{\sigma}_z\left(\quad\right)$$

$$=\frac{\hbar}{2}\quad\cos^2\theta\langle\uparrow n|e^{iE_\alpha t}+\sin\theta\cos\theta\langle\downarrow n+1|e^{iE_\alpha t}\Big\}$$

43.4

$$|\uparrow, n(t)\rangle = \cos\theta\, e^{-iE_\alpha t/\hbar}\, |\alpha(0)\rangle$$
$$+ \sin\theta\, e^{-iE_\beta t/\hbar}\, |\beta(0)\rangle$$

$$= \cos^2\theta\, e^{-iE_\alpha t/\hbar}\, |\uparrow, n(0)\rangle$$
$$+ \cos\theta \sin\theta\, e^{-iE_\alpha t/\hbar}\, |\downarrow, n+1, (0)\rangle$$
$$+ \sin^2\theta\, e^{-iE_\beta t/\hbar}\, |\uparrow, n(0)\rangle$$
$$- \cos\theta \sin\theta\, e^{-iE_\beta t/\hbar}\, |\downarrow, n+1, 0\rangle$$

$$= \left(\cos^2\theta\, e^{-iE_\alpha t/\hbar} + \sin^2\theta\, e^{-iE_\beta t/\hbar}\right) |\uparrow, n\rangle$$
$$+ \left(\cos\theta \sin\theta\, e^{-iE_\alpha t/\hbar} - e^{-iE_\beta t/\hbar}\right) |\downarrow, n+1\rangle$$

$$\cancel{\langle S_z\rangle} = \left(\overset{\frac{1+\cos 2\theta}{2}}{\cos^2\theta}\, e^{-i\lambda t} + \overset{\frac{1-\cos 2\theta}{2}}{\sin^2\theta}\, e^{i\lambda t}\right) |\uparrow, n\rangle$$
$$+ \cos\theta \sin\theta \left(e^{-i\lambda t} - e^{+i\lambda t}\right) |\downarrow, n+1\rangle$$

$$= \cancel{\tfrac{1}{2}}\, \cos\lambda t - i\left(\cos 2\theta + \sin 2\theta\right) \sin\lambda t$$

resonance $\lambda = g\sqrt{n+1}, \quad \theta = \frac{\pi}{4}$

$\langle S_z\rangle$

43.5

$$|\uparrow, n(t)\rangle = \left(\cos\lambda t + \frac{\cos 2\theta}{2}\left(e^{-i\lambda t} + e^{i\lambda t}\right)\right)|\uparrow, n(0)\rangle$$

$$\left(\frac{\sin 2\theta}{2}\left(e^{-i\lambda t} - e^{i\lambda t}\right)\right)|\downarrow, n+1; (0)\rangle$$

$\theta = \frac{\pi}{4}$; $\langle S_z\rangle = \frac{\hbar}{2}\left(\cos^2\lambda t - \sin^2\lambda t\right)$; $\lambda = g\sqrt{n+1}$ $= \Omega$ at resonance.