Lecture 21

Eigen States of a 3d quantum rotator are characterized by two quantum numbers: value of $\hat{\vec{L}}^2$ and $\hat{L}_z$

Any other state can be represented as

$$|\psi\rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{l,m} \, |l;m\rangle.$$

The eigen states $|l;m\rangle$, which were introduced from the operator algebra can be found as functions of polar coordinates $(\theta;\varphi)$:

$$|l;m\rangle \Leftrightarrow f_l^m(\theta,\varphi) = f_l^m(\theta,\varphi)$$

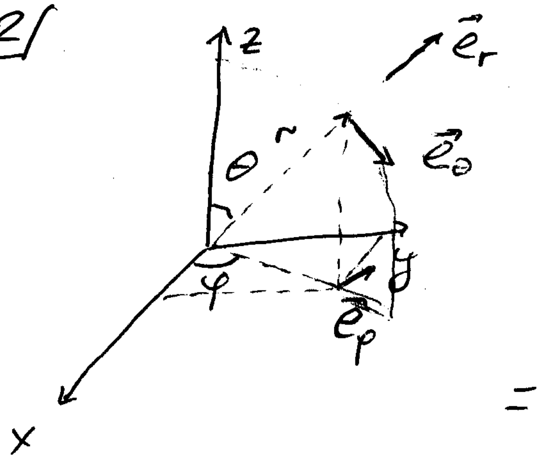
To find an explicit form of $\psi(\theta,\varphi)$, we have to represent operators $\hat{\vec{L}}^2$ & L_z in the angular coordinates' form:

The momentum operator

$$\hat{\vec{p}} = -i\hbar\nabla = -i\hbar\left[\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right],$$

where $\vec{e}_r$; $\vec{e}_\theta$ and $\vec{e}_\varphi$ are unit vectors

21.2



$$\hat{\vec L} = \vec r \times \hat{\vec p} =$$

$$= -i\hbar\left[r\,\cancel{(\vec e_r \times \vec e_r)}^{\,0}\frac{\partial}{\partial r} + [\vec e_r \times \vec e_\theta]\frac{\partial}{\partial \theta} + \vec e_r \times \vec e_\varphi \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\right)$$

$$= -i\hbar\left([\vec e_r \times \vec e_\theta]\frac{\partial}{\partial \theta} + [\vec e_r \times \vec e_\varphi]\frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\right)$$

$$= -i\hbar\left(\vec e_\varphi \frac{\partial}{\partial \theta} + (-\vec e_\theta)\frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\right)$$

The unit vectors can be resolved in their cartesian components:

$\cancel{\vec e_\theta}$

$$\vec e_\theta = (\cos\theta\cos\varphi)\vec e_x + (\cos\theta\sin\varphi)\vec e_y + \sin\theta\,\vec e_z$$

$$\vec e_\varphi = -\sin\varphi\,\vec e_x + \cos\varphi\,\vec e_y$$

$$\hat L_x = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\,\mathrm{cotan}\,\theta\frac{\partial}{\partial\varphi}\right)$$

$$\hat L_y = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\,\mathrm{cotan}\,\theta\frac{\partial}{\partial\varphi}\right)$$

$$\hat L_z = -i\hbar\frac{\partial}{\partial\varphi} \quad (*)$$

21.3 To describe $|\ell, m\rangle$, we need

$$\hat{L}_z \text{ and } \hat{\vec{L}}^2 = \hat{L}_+ \hat{L}_- + \hat{L}_z^2 - \hbar \hat{L}_z,$$

and we can construct

$$\hat{L}_\pm = \hat{L}_x \pm i \hat{L}_y :$$

$$\hat{L}_\pm = \pm \hbar e^{\pm i\varphi} \left(\frac{\partial}{\partial \theta} \pm i \operatorname{cotan}\theta \frac{\partial}{\partial \varphi} \right)$$

$$\hat{L}_+ \cdot \hat{L}_- f = -\hbar^2 e^{+i\varphi} \left(\frac{\partial}{\partial \theta} + i \operatorname{cotan}\theta \frac{\partial}{\partial \varphi} \right) \left(e^{-i\varphi} \left[\frac{\partial}{\partial \theta} - i \operatorname{cotan}\theta \frac{\partial}{\partial \varphi} \right] f \right)$$

$$\left(\frac{\cos\theta}{\sin\theta} \right)' = \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}$$

$$= -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + i \frac{1}{\sin^2\theta} \frac{\partial}{\partial \varphi} \cancel{- i \operatorname{cotan}\theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi}} \right.$$

$$+ (-i) i \operatorname{cotan}\theta \frac{\partial}{\partial \theta} + (-i) \operatorname{cotan}^2\theta \frac{\partial}{\partial \varphi} + \operatorname{cotan}^2\theta \frac{\partial^2}{\partial \varphi^2} +$$

$$\left. \cancel{+ i \operatorname{cotan}\theta \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \theta}} \right) f$$

$$= -\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} + i \frac{\sin^2\theta}{\sin^2\theta} \frac{\partial}{\partial \varphi} + \operatorname{cotan}^2\theta \frac{\partial^2}{\partial \varphi^2} \right] f$$

$$\hat{\vec{L}}^2 = -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right)$$

21.4)

Let us look for eigen functions:

$$\hat{L}_z f_\ell^m(\theta,\varphi) = \hbar m f_\ell^m(\theta,\varphi) \qquad (1)$$

$$\hat{L}^2 f_\ell^m(\theta,\varphi) = \hbar^2 \ell(\ell+1) f_\ell^m(\theta,\varphi) \qquad (2)$$

A solution for (1) is simple:

$$-i\hbar\frac{\partial}{\partial\varphi} f_\ell^m(\theta,\varphi) = \hbar m f_\ell^m(\theta,\varphi)$$

$$f_\ell^m(\theta,\varphi) = y_\ell^m(\theta) e^{im\varphi}.$$

For integer k, m is also integer and

$$f_\ell^m(\theta,\varphi+2\pi) = f_\ell^m(\theta,\varphi).$$

However, for $\ell = \frac{2k+1}{2}$, $k = 0; 1; 2\dots$

$$f_\ell^m(\theta,\varphi+2\pi) = f_\ell^m(\theta,\varphi) e^{i\pi} = -f_\ell^m(\theta,\varphi)$$

These states do not describe states of angular momentum. Still, the corresponding operators describe "internal" properties of elementary particles.

21.5 / Now, we can obtain the equation for the polar component $\mathcal{Y}_\ell^m(\theta)$

$$0 = \sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\mathcal{Y}_\ell^m(\theta)}{d\theta}\right) + \left(\ell(\ell+1)\sin^2\theta - m^2\right)\mathcal{Y}_\ell^m(\theta)$$

The solution is the associated Legendre function

$$P_\ell^m(x) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|} P_\ell(x),$$

and $P_\ell(x)$ is the ℓth Legendre polynomial

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell .$$

The normalized angular harmonics are:

$$\psi_\ell^m(\theta,\varphi) = \mathcal{Y}_\ell^m(\theta,\varphi) =$$

$$= \epsilon\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\, e^{im\varphi} P_\ell^m(\cos\theta).$$

$\epsilon = 1$ for $m \le 0$ and $\epsilon = (-1)^m$ for $m > 0$,

introduced for $\mathcal{Y}_\ell^{-m} = (-1)^m (\mathcal{Y}_\ell^m)^*$