Lecture 19

The generalized uncertainty principle

$$\sigma_A^2 \cdot \sigma_B^2 \geq \left(\frac{1}{2i}\langle[A,B]\rangle\right)^2$$

$$\sigma_A \stackrel{def}{=} \langle\psi|(\hat{A}-\langle\hat{A}\rangle)^2|\psi\rangle$$

$$= \langle(\hat{A}-\langle A\rangle)\psi|(\hat{A}-\langle\hat{A}\rangle)\psi\rangle$$

$$= \langle f|f\rangle$$

where $|f\rangle = (\hat{A}-\langle\hat{A}\rangle)|\psi\rangle$

$\sigma_B = \langle g|g\rangle$, with $|g\rangle = (\hat{B}-\langle\hat{B}\rangle)|\psi\rangle$

$$\sigma_A \cdot \sigma_B = \langle f|f\rangle \cdot \langle g|g\rangle$$

$$\geq \langle f|g\rangle \cdot \langle f|g\rangle^*$$

$$= (\mathrm{Re}\,\langle f|g\rangle)^2 + (\mathrm{Im}\,\langle f|g\rangle)^2$$

$$\geq (\mathrm{Im}\,\langle f|g\rangle)^2$$

19.2

$$\mathrm{Im}\langle f|g\rangle = \frac{\langle f|g\rangle - \langle g|f\rangle}{2i}$$

$$\langle f|g\rangle = \langle (\hat{A} - \langle \hat{A}\rangle)\psi | (\hat{B} - \langle \hat{B}\rangle)\psi\rangle$$

$$= \langle \psi | (\hat{A} - \langle \hat{A}\rangle)(\hat{B} - \langle \hat{B}\rangle) | \psi\rangle$$

$$= \langle \psi | \hat{A}\cdot\hat{B} | \psi\rangle - \langle \hat{A}\rangle\langle \hat{B}\rangle$$

$$\langle g|f\rangle = \langle \psi | \hat{B}\hat{A} | \psi\rangle - \langle A\rangle\langle B\rangle$$

$$\mathrm{Im}\langle f|g\rangle = \frac{1}{2i}\langle \psi | [\hat{A}, \hat{B}] | \psi\rangle$$

and

$$\sigma_A \cdot \sigma_B \geq \left(\frac{1}{2i}\langle \psi | [\hat{A}, \hat{B}] | \psi\rangle\right)^2.$$

19.31 Next topic: angular momentum

Griffiths, Section 4.3.

In classical mechanics

$$\vec{L} = \vec{r} \times \vec{p}.$$

In quantum mechanics $\hat{p} = -i\hbar(\partial_x; \partial_y; \partial_z)$

$\partial_x = \frac{\partial}{\partial x}$

$$\hat{L}_x = y(-i\hbar\partial_z) - z(-i\hbar\partial_y)$$

Let us consider the commutator

$$[\hat{L}_x; \hat{L}_y] = [y\hat{p}_z - z\hat{p}_y; z\hat{p}_x - x\hat{p}_z]$$

$$= [y\hat{p}_z; z\hat{p}_x] - \cancel{[z\hat{p}_y; z\hat{p}_x]}^{\,0} - \cancel{[y\hat{p}_z; x\hat{p}_z]}^{\,0} +$$

$$+ [z\hat{p}_y; x\hat{p}_z]$$

$$= y p_x \cdot [\hat{p}_z; z] + x\hat{p}_y [z; \hat{p}_z] = -i\hbar\, y p_x + i\hbar x p_y$$

$$= i\hbar(x\hat{p}_y - y\hat{p}_x) = i\hbar L_z.$$

19.4)

$$[\hat{L}_x ; \hat{L}_y] = i\hbar \hat{L}_z$$
$$[\hat{L}_y ; \hat{L}_z] = i\hbar \hat{L}_x$$
$$[\hat{L}_z ; \hat{L}_x] = i\hbar \hat{L}_y$$

$$\sigma_x^2 \cdot \sigma_y^2 \geq \left(\frac{1}{2i} i\hbar \langle \hat{L}_z \rangle\right)^2 = \frac{\hbar^2}{4} \langle L_z \rangle^2$$

what if $\hat{L}_x |\psi\rangle = L_x |\psi\rangle$? $\sigma_x = 0$.

The total angular momentum commutes with $\hat{L}_x ; \hat{L}_y ; \hat{L}_z$, if tot. ang. mom. is defined ~~as~~

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + L_z^2.$$

Indeed $[\hat{L}^2 ; \hat{L}_x] = [\hat{L}_x^2 ; \hat{L}_x] + [\hat{L}_y^2 ; L_x] + [\hat{L}_z^2 ; \hat{L}_x]$

$$= \hat{L}_y [\hat{L}_y ; \hat{L}_x] + [\hat{L}_y ; \hat{L}_x] \hat{L}_y + \hat{L}_z [\hat{L}_z ; \hat{L}_x] + [\hat{L}_z ; \hat{L}_x] \hat{L}_z$$

$$= \cancel{\hat{L}_y (-i\hbar \hat{L}_z)} + \cancel{(-i\hbar \hat{L}_z \hat{L}_y)} + \cancel{(i\hbar \hat{L}_z \hat{L}_y)} + \cancel{(i\hbar \hat{L}_y \hat{L}_z)}$$

$= 0$

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B} ; \hat{C}] + [\hat{A} ; \hat{C}]\hat{B}$$ (prove at home)

no proof in class

19.5

$$[\hat{L}^2; \hat{L}_x] = [\hat{L}^2; \hat{L}_y] = [\hat{L}^2; \hat{L}_z] = 0.$$

We introduce $L_\pm = L_x \pm i L_y$

$$[\hat{L}^2; \hat{L}_\pm] = 0; \qquad [L_z, L_\pm] = [\hat{L}_z, \hat{L}_x] \pm i[\hat{L}_z, \hat{L}_y]$$

$$= i\hbar \hat{L}_y \pm i(-i\hbar \hat{L}_x) =$$

$$= \pm\hbar \hat{L}_x + i\hbar \hat{L}_y = \pm\hbar \hat{L}_\pm$$

$$[\hat{L}_x; \hat{L}_\pm] = [\hat{L}_x; \hat{L}_x] \pm i[\hat{L}_x; \hat{L}_y]$$

$$= \pm i \cdot i\hbar \hat{L}_z = \mp \hbar \hat{L}_z.$$

$$[\hat{L}_y, \hat{L}_\pm] = [L_y; L_x] = -i\hbar L_z$$

$$[\hat{L}_-; \hat{L}_+] = -\hbar \hat{L}_z - i \cdot (-i\hbar L_z) = -2\hbar \hat{L}_z.$$

$$\hat{L}_z \hat{L}_\pm = \hat{L}_\pm(\hat{L}_z \pm \hbar)$$ is another form of the commutation relation.

19.6

If $|f\rangle$ is an eigen state of $\hat{L}_z$, i.e. $\hat{L}_z|f\rangle = \mu \cdot |f\rangle$, then

$L_\pm |f\rangle$ is an eigen state of L_z as well. The eigen value is given by $(\mu \pm \hbar)$.

$$\hat{L}_z \circ \hat{L}_\pm |f\rangle = \hat{L}_\pm (\hat{L}_z \pm \hbar)|f\rangle =$$

$$= \hat{L}_\pm |f\rangle \cdot (\mu \pm \hbar), \quad \text{q.e.d.}$$

If $|f\rangle$ is an eigen state of $\hat{L}^2$, i.e. $\hat{L}^2|f\rangle = \lambda |f\rangle$, then $L_\pm |f\rangle$ is also an eigen state of $\hat{L}^2$ with the same λ.

$$\hat{L}^2 \cdot \hat{L}_\pm |f\rangle = \hat{L}_\pm \hat{L}^2 |f\rangle = \hat{L}_\pm |f\rangle \cdot \lambda.$$

We can characterize states by λ and μ: $|f\rangle = |\lambda; \mu\rangle$,

$$\hat{L}^2 |\lambda; \mu\rangle = \lambda^2 |\lambda, \mu\rangle \quad \hat{L}_z |\lambda, \mu\rangle = \mu |\lambda, \mu\rangle$$

19.B/

$\cancel{\lambda \geq \mu^2}$ However, λ and μ are not arbitrary.

$$\cancel{\langle \lambda,\mu | \lambda | \lambda,\mu \rangle} \quad \lambda = \langle \lambda,\mu | \hat{L}^2 | \lambda,\mu \rangle$$

$$= \underbrace{\langle \lambda,\mu | L_x^2 | \lambda,\mu \rangle}_{\geq 0} + \underbrace{\langle \lambda,\mu | \hat{L}_y^2 | \lambda,\mu \rangle}_{\text{observable} \; \langle L_y(\lambda\mu) | L_y \lambda\mu \rangle \geq 0} + \underbrace{\langle \lambda,\mu | \hat{L}_z^2 | \lambda,\mu \rangle}_{=\mu^2}$$

$$\geq \mu^2 \qquad \boxed{\lambda \geq \mu^2}$$

If we apply $\hat{L}_+$ to $|\lambda,\mu\rangle$ we obtain $\hat{L}_+^n |\lambda,\mu\rangle = |\lambda; \mu + \hbar n\rangle$ and there is a "top" state with $\mu = \hbar l$.

$$\hat{L}^2 = L_x^2 + L_y^2 + L_z^2 = (\hat{L}_\pm \cdot \hat{L}_\mp) \mp \hbar L_z + L_z^2$$

$$L^2 |\lambda; \hbar l\rangle = \hbar^2 (l+1) l |\lambda; \hbar l\rangle, \text{ or } \lambda = \hbar^2 l(l+1)$$

since $L_+ |\lambda; \hbar l\rangle = 0$.

For "bottom state"

$$L^2 |\lambda; \hbar \bar{l}\rangle = (-\hbar^2 \bar{l} + \hbar^2 \bar{l}^2) |\lambda; \hbar\bar{l}\rangle = \hbar^2 \bar{l}(\bar{l} - 1) |\lambda; \hbar \bar{l}\rangle$$

$$\lambda = \hbar^2 \bar{l}(\bar{l} - 1)$$

19.8

$$\hbar^2 l(l+1) = \hbar^2(\bar{l}-1)\bar{l}$$

has two solutions

$$l = -\bar{l} \quad \text{and} \quad l = \bar{l}+1,$$

but $l > \bar{l}$, so only $l = -\bar{l}$

has a meaning.

Let us apply $\hat{L}_+^n$ to $|\hbar^2 l(l+1); -\hbar l\rangle$,

$$\hat{L}_+^n |\hbar^2 l(l+1); -\hbar l\rangle = |\hbar^2 l(l+1); -\hbar l + \hbar n\rangle$$

$$= |\hbar^2 l(l+1); \hbar l\rangle$$

for $n = 2l$

and $l = \frac{n}{2}$, where n is integer.

μ has $(n+1)$ states, i.e. $(2l+1)$ states.

For angular momentum, however, only $l \in$ integer number are allowed, but the algebra of operators L_x, L_y, L_z allows also half integer values, but do not originate from real-space rotations