Lecture 17

$$\psi(x,t) = \sum c_n \psi_n(x)\, e^{-iE_n t/\hbar}$$ and for

every $\psi(x,t)$ we have a corresponding set of c_n, written in basis $\psi_n(x)$

$$|\psi\rangle = \sum c_n |n\rangle_t = \sum c_n e^{-iE_n t/\hbar} |n\rangle$$

$$|\psi\rangle \Leftrightarrow \{c_n\}$$

The product of

$\psi(x,t)$ and $\varphi(x,t)$ is defined as

$$\int \varphi^*(x,t)\, \psi(x,t)\, dx$$

$$= \sum_{n,m} d_n c_m \int \psi_n^* \psi_m \, dx = \sum_n d_n c_n$$

$$= \langle \varphi | \psi \rangle = \{d_1 \ldots d_n \ldots\} \begin{pmatrix} c_1 \\ \vdots \\ c_n \\ \vdots \end{pmatrix}.$$

The integral Schwarz inequality:

$$\left| \int_{-\infty}^{\infty} f^*(x)\, g(x)\, dx \right| \leq \sqrt{\int |f(x)|^2 dx \int |g(x)|^2 dx}.$$

Pb. 2)

The expectation value of an observable Q:

$$\langle Q(t)\rangle = \int \psi^*_{(x,t)} \hat{Q}\, \psi(x,t)\, dx$$

Observables are real numbers

$$\langle Q\rangle = \langle Q\rangle^*$$

$$\langle \psi | \hat{Q} | \psi\rangle = \int \psi(x,t) \left(\hat{Q}\psi(x,t)\right)^* dx$$

$$= \langle \hat{Q}\psi | \psi\rangle,$$

But $\langle \hat{Q}\psi|$ is related to $\hat{Q}|\psi\rangle$

by a Hermitian conjugation:

$$\hat{Q}\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\\ \vdots\end{bmatrix} = \hat{Q}|\psi\rangle = \begin{bmatrix}c_1\\ \vdots\\ c_n\\ \vdots\end{bmatrix} \qquad \langle Q\psi| = [c_1^* \ldots\ldots c_n^* \ldots]$$

$$= \hat{Q}^+ \cdot [\alpha_1^* \ldots \alpha_n^* \ldots]\, \hat{Q}^+$$

$$\hat{Q}^+ = (Q^*)^T$$

Consider $|\psi\rangle = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$, $\hat{Q} = \begin{pmatrix} q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}$

$$\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \hat{Q}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix} q_{11}\alpha_1 + q_{12}\alpha_2\\ q_{21}\alpha_1 + q_{22}\alpha_2\end{pmatrix} \quad (c_1^*\ c_2^*)\begin{pmatrix} \\ \end{pmatrix} = \alpha_1^*\alpha_1 q_{11} + \alpha_1^* q_{12}\alpha_2$$
$$+ \alpha_2^* q_{21}\alpha_1 + q_{22}\alpha_2\alpha_2^*$$

$$(c_1^*\ \ c_2^*) = (\alpha_1^*\ \ \alpha_2^*)\begin{pmatrix}\tilde{q}_{11} & \tilde{q}_{12}\\ \tilde{q}_{22} & \tilde{q}_{22}\end{pmatrix} = \left(\alpha_1^*\tilde{q}_{11} + \alpha_2^*\tilde{q}_{21},\ \alpha_1^*\tilde{q}_{12} + \alpha_2^*\tilde{q}_{22}\right)$$

P.3

$$\left(q_{11}^* \alpha_1^* + q_{12}^* \alpha_2^*\right) = \alpha_1^* \tilde{q}_{11} + \alpha_2^* \tilde{q}_{21}$$

$$\left(q_{21}^* \alpha_1^* + q_{22}^* \alpha_2^*\right) = \alpha_1^* \tilde{q}_{12} + q_2^* \tilde{q}_{22}$$

$$q_{11}^* = \tilde{q}_{11} \qquad q_{21}^* = \tilde{q}_{12}$$

$$q_{12}^* = \tilde{q}_{21} \qquad q_{22}^* = \tilde{q}_{22}$$

$$Q^+ = \begin{pmatrix} \tilde{q}_{11} & \tilde{q}_{12} \\ \tilde{q}_{21} & \tilde{q}_{22} \end{pmatrix} = \begin{pmatrix} q_{11}^* & q_{21}^* \\ q_{12}^* & q_{22}^* \end{pmatrix}$$

For observables

$\langle Q \rangle \in \mathrm{Re}$

$$\langle Q\psi | \psi \rangle = \langle \hat{\psi} | \hat{Q} \psi \rangle \overset{def}{=} \langle Q^+ \psi | \psi \rangle$$

$$\hat{Q} = \hat{Q}^+.$$

Eigenvalues of Q are real;

Take $|\psi_Q\rangle = |\text{eigenfunction of } \hat{Q}\rangle$

$$\hat{Q}|\psi_Q\rangle = q_Q |\psi_Q\rangle \quad \text{and} \quad \langle \psi_Q | \hat{Q} | \psi_Q \rangle = q_Q = q_Q^*.$$

17.4) ~~Lecture 18~~ Lecture 8

$$\sigma_Q^2 = \langle \psi | (\hat{Q} - \langle \hat{Q} \rangle)^2 | \psi \rangle =$$

$$\langle \psi | (\hat{Q} - \langle Q \rangle)(\hat{Q} - \langle Q \rangle) | \psi \rangle$$

$\cancel{\langle \psi_n Q \psi_n}$ Q_n $\langle Q \rangle = \langle \psi_n | \hat{Q} | \psi_n \rangle = Q_n \cdot 1$

$$\langle \psi | (\hat{Q} - \langle \hat{Q} \rangle)(Q_n - \langle Q_n \rangle) | \psi \rangle = 0.$$

Example that we studied are stationary states $\equiv$ eigen states of a Hamiltonian.

$$\hat{H} \psi_n = E_n \psi_n .$$

<u>Theorem</u>. Eigenfunctions belonging to distinct eigenvalues are orthogonal.

Suppose $\hat{Q} f_{(x)} = q_1 f_{(x)}$; $\hat{Q} g_{(x)} = q_2 g_{(x)}$

$\hat{Q} | f \rangle = q_1 | f \rangle$ $\hat{Q} | g \rangle = q_2 | g \rangle$

$$q_2 \langle f | g \rangle = \langle f | Q g \rangle = \langle \hat{Q}^+ f | g \rangle = \langle Q f | g \rangle = \langle f | g \rangle q_1$$

$$\langle f | g \rangle (q_1 - q_2) = 0 \Rightarrow \langle f | g \rangle = 0.$$

$(\alpha_1^* q_{12} \alpha_2) + (\alpha_2^* q_{21} \alpha_1)$

17.3

Proof of the Schwarz inequality:

$$|\langle\alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle \cdot \langle\beta|\beta\rangle$$

Consider $|\gamma\rangle = |\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle$

$\langle\gamma|\gamma\rangle \geq 0 \quad \Leftarrow \int |\psi(x)|^2 dx$

$$\left(\langle\beta| - \frac{\overset{\langle\alpha|\beta\rangle^*}{\overset{\parallel}{\langle\beta|\alpha\rangle^*}}}{\langle\alpha|\alpha\rangle}\langle\alpha|\right)\left(|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle\right)$$

$$= \langle\beta|\beta\rangle - \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle} - \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle}$$

$$+ \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle^2} \cdot \langle\alpha|\alpha\rangle \geq 0$$

$$\langle\beta|\beta\rangle \cdot \langle\alpha|\alpha\rangle \geq \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle$$

We need a Schwarz inequality for to proof the Generalized uncertainty principle.

$$\sigma_A^2 \cdot \sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle\right)^2$$

If two operators do not commute, are incompatible, i.e. a quantum system cannot be in a definite state for both of them operators

$\boxed{17.6}$

Compatible observables do admit complete sets of simultaneous eigenfunctions.

<u>Proof</u>

$$\sigma_A^2 \overset{def}{=} \langle \psi | (\hat{A} - \langle A \rangle)^2 | \psi \rangle =$$

$$= \langle (\hat{A} - \langle \hat{A} \rangle)\psi | (\hat{A} - \langle \hat{A} \rangle)\psi \rangle$$

$$= \langle f | f \rangle, \text{ where } |f\rangle = (\hat{A} - \langle A \rangle)|\psi\rangle$$

Similarly

$$\sigma_B^2 = \langle \psi | (\hat{B} - \langle B \rangle)^2 | \psi \rangle =$$

$$= \langle (B - \langle B \rangle)\psi | (\hat{B} - \langle B \rangle)\psi \rangle = \langle g | g \rangle$$

$$\sigma_A^2 \cdot \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle$$

$$\geq \langle f | g \rangle \langle g | f \rangle$$

$$\geq \left| \frac{1}{2i} \left(\langle f | g \rangle - \langle g | f \rangle \right) \right|^2$$

17.7

$$\langle f|g\rangle = \langle(\hat{A}-\langle A\rangle)\psi|\hat{B}-\langle B\rangle|\psi\rangle$$

$$= \langle\psi|(\hat{A}-\langle\hat{A}\rangle)(\hat{B}-\langle\hat{B}\rangle)|\psi\rangle$$

$$= \langle\psi|\hat{A}\hat{B}|\psi\rangle - \langle\psi|A|\psi\rangle\langle\psi|B|\psi\rangle$$

$$\langle g|f\rangle = \langle\psi|BA|\psi - \langle\psi|B|\psi\rangle\langle\psi|A|\psi\rangle$$

$$\sigma_A^2\sigma_B^2 \geq \left[\frac{1}{2i}\langle\psi|[\hat{A},\hat{B}]\psi\rangle\right]^2$$

$$[\hat{A},\hat{B}] = \hat{A}\hat{B}-\hat{B}\hat{A},$$

$$[\hat{A},\hat{B}]^+ = [\hat{A}\hat{B}]^+ - [\hat{B}\hat{A}]^+ = \hat{B}^+\hat{A}^+ - \hat{A}^+\hat{B}^+ =$$

$$= \hat{B}\hat{A} - \hat{A}\hat{B} = -[\hat{A},\hat{B}]$$

$$\sigma_x^2\cdot\sigma_p^2 \geq \frac{\hbar^2}{4}. \qquad [\hat{p},\hat{x}] = -i\hbar$$

just a number!

17.8/ Another famous example, but often misinterpreted!

$\hat{A} = \hat{H}; \quad \hat{B} = \hat{Q}$

$$\sigma_H^2 \cdot \sigma_Q^2 \geq \left(\frac{i}{2}\langle[\hat{Q},\hat{H}]\rangle\right)^2$$

But $\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \cancel{\left\langle\frac{\partial Q}{\partial t}\right\rangle} \to 0$ for most operators.

$$\sigma_H^2 \cdot \sigma_Q^2 \geq \frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right|$$

$$\sigma_E \cdot \frac{\sigma_Q}{\frac{d\langle Q\rangle}{dt}} \geq \frac{\hbar}{2} = \sigma_E \cdot \Delta t \geq \frac{\hbar}{2},$$

where $\frac{\sigma_Q}{\frac{d\langle Q\rangle}{dt}} = \Delta t$

Δt is the time for $\langle Q\rangle$ to change by σ_Q

$\frac{d\langle x\rangle}{dt} = \frac{p}{m}$ (Lecture 3?) σ_x

$\sigma_H = \left|\frac{dE}{dp}\right|\sigma_p = \frac{p}{m}\cdot\sigma_p$

$\Delta E \cdot \Delta t =$

$= \frac{p}{m}\sigma_p \cdot \sigma_x \frac{m}{p} \geq \frac{\hbar}{2}.$

17.9

If a system in an eigen state $|\psi_n\rangle$

$$|\psi_n(t)\rangle = e^{-\frac{iE_n t}{\hbar}} |\psi_n(0)\rangle$$

$$\langle\psi_n(t)|Q|\psi_n(t)\rangle = \langle\psi_n(0)|Q|\psi_n(0)\rangle \equiv \text{const.}$$

$$\frac{dQ}{dt} = 0.$$