Lecture 14. Propagation through a potential barrier.

We introduced the probability current

$$j = -\frac{i\hbar}{2m}\left[\psi^*(x)\frac{d}{dx}\psi(x) - \psi(x)\frac{d}{dx}\psi^*(x)\right],$$

which satisfies the continuity equation:

$$\frac{\partial P}{\partial t} + \operatorname{div} j = 0.$$

Consider $\psi(x) = e^{i\varphi(x)} \cdot \sqrt{P(x)}$.

$$j = -\frac{i\hbar}{2m}\Big[i\varphi'(x)\cdot P(x) - (-i\varphi'(x))P(x) \quad \leftarrow \text{diff. } e^{i\varphi(x)}$$

$$\sqrt{P(x)}\cdot\frac{P'(x)}{2\sqrt{P(x)}} - \sqrt{P(x)}\cdot\frac{P'(x)}{2\sqrt{P(x)}}\Big] \leftarrow \text{diff } \sqrt{P(x)}$$

$$= \frac{\hbar}{m}\varphi'(x)\cdot P(x)$$ and does <u>not</u> contain $\frac{dP(x)}{dx}$!

14.2)

For $\psi(x) = e^{ipx/\hbar} \cdot c(x)$ we obtain the probability current

$$j(x) = \frac{\hbar}{m} \cdot \frac{p}{\hbar} c^2(x) = \frac{p}{m} c^2(x) = v_0 c^2(x)$$

For an arbitrary potential $U(x)$ with $U(x) \underset{x\to\pm\infty}{=} \cancel{U_0} = \text{const}$, we can define the probability current at $x = \pm\infty$.

Indeed, the solution of the Schrödinger equation is $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = (E - U_\infty)\psi(x)$

$$\psi_\pm(x) = c_\pm \cdot e^{\pm i\sqrt{\frac{2m(E-U_\infty)}{\hbar^2}}\, x}.$$

$$j_\pm(x) = \pm v(x)\, c^2(x).$$

In many problems the solution for $\cancel{x} \to +\infty$ is ~~written~~ assumed to be given by

$\psi(x) = C_+ e^{i\sqrt{\frac{2m(E-U_\infty)}{\hbar^2}}\, x}$, i.e. there is only flux to the $+x$ direction, and no flux from $x \to +\infty$ to ~~x→0~~ smaller x.

14.3 Solving the Schrödinger equation, we obtain for $x \to -\infty$ the following expression

$$\psi_{-\infty}(x) = C_{\pm}^{(-\infty)} e^{\pm i \sqrt{\frac{2m(E-U_{-\infty})}{\hbar^2}}}$$

and
$$j_{+}(-\infty) = v(-\infty)\left|C_{+}^{(-\infty)}\right|^2$$
$$j_{-}(-\infty) = v(-\infty)\left|C_{-}^{(-\infty)}\right|^2$$

We can say that $j_{+}(-\infty) = j_{in}$ is the ~~flux of~~ incoming flux of the probability for a particle;

$j_{-}(-\infty) = j_r$ is the reflected current density;

$j_{+}(+\infty) = j_t$ is the transmitted flux.

$T = \frac{j_t}{j_{in}}$ is the transmission coefficient.

$R = \frac{j_r}{j_{in}}$ is the reflection coefficient.

14.4

Note, that because $\frac{\partial \rho}{\partial t} = 0$ $\quad \mathrm{div}\, j = \frac{\partial j}{\partial x} \equiv 0$,

and $j_{in} - j_r = j_t$. We have $T + R = 1$

$$j_t + j_r = j_{in}$$

$$T = \frac{v(+\infty)}{v(-\infty)} \frac{|C_+^{(+\infty)}|^2}{|C_+^{(-\infty)}|^2} = \sqrt{\frac{E - U_r}{E - U_l}} \cdot \left| \frac{C_+^{(+\infty)}}{C_+^{(-\infty)}} \right|^2$$

$$R = \left| \frac{C_-^{(-\infty)}}{C_+^{(-\infty)}} \right|^2 .$$

Reflection coefficients are identical for right and left movers.

$$\psi = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} \qquad x \to -\infty$$
$$\psi = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x} \qquad x \to +\infty$$

Due to linearity of the Sch. equation,

$$A_2 = \alpha A_1 + \beta B_1$$

For stationary Sch. equation

$\psi^* = A_2^* e^{-ik_2 x} + B_2^* e^{ik_2 x}$ is also a solution

$$B_2^* = \alpha B_1^* + \beta A_1^* \quad \text{or} \quad B_2 = \alpha^* B_1 + \beta^* A_1$$
$$A_2 = \alpha A_1 + \beta B_1$$

$B_2 = 0$ for right movers; $\frac{B_1}{A_1} = -\frac{\beta^*}{\alpha^*}$ $\quad R = \left|\frac{\beta}{\alpha}\right|^2$

Equal

$A_1 = 0$ for left movers $\frac{A_2}{B_2} = \frac{\beta}{\alpha^*}$ $\quad R = \left|\frac{\beta}{\alpha^*}\right|^2$