Lecture 13.

We introduced operators

a_+ and a_- :

$$a_\pm = \frac{1}{\sqrt{2}}\left(\mp \frac{d}{d\xi} + \xi\right) \quad \text{or} \quad a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$$

The Hamiltonian can be rewritten as a bilinear combination of a_+ & a_-:

$$\left(-\frac{d^2}{d\xi^2} + \xi^2\right)\psi_n(\xi) = K_n\,\psi_n(\xi), \quad \text{where} \quad K_n = \frac{2E_n}{\hbar\omega}$$

$$\left(2a_+a_- + 1\right)\psi_n(\xi) = K_n\,\psi_n(\xi),$$ or in original units ~~coordinates/momenta~~

$$\hbar\omega\left(a_+a_- + 1/2\right)\psi_n(\xi) = E_n\,\psi_n(\xi).$$

Operators a_+ & a_- have an important property:

If ψ_n is an eigen state ψ_n with energy E_n,

then 1. $a_+\psi_n(\xi)$ is also an eigen state with energy $E_n + \hbar\omega$

2. $a_-\psi_n(\xi)$ is an eigen state with energy $E_n - \hbar\omega$

13.2 **Proof:** is based on the following commutation relation $[\hat{a}_-, \hat{a}_+] = \hat{1}$.

Indeed, $a_- a_+ \psi - a_+ a_- \psi =$

$$= \frac{1}{2}\left[\left(\frac{d}{d\xi}+\xi\right)\left(-\frac{d}{d\xi}+\xi\right)-\left(-\frac{d}{d\xi}+\xi\right)\left(\frac{d}{d\xi}+\xi\right)\right]\psi$$

$$= \frac{1}{2}\left[2\frac{d}{d\xi}(\xi\psi) - 2\xi\frac{d}{d\xi}\psi\right] = \hat{1}\psi$$

Consider $a_+\psi_n$ for $\hbar\omega(a_+a_- + 1/2)\psi_n = E_n\psi_n$.

We have $\hbar\omega(a_+a_- + 1/2)(a_+\psi_n) =$

$$= \hbar\omega\left[a_+a_-a_+ + 1/2\, a_+\right]\psi_n$$

$$= \hbar\omega\left[a_+a_+a_- + a_+ + 1/2\, a_+\right]\psi_n$$

$$= \hbar\omega\, a_+\left[a_+a_- + 1/2 + 1\right]\psi_n$$

$$= a_+(E_n + \hbar\omega)\psi_n = (E_n + \hbar\omega)(a_+\psi_n),$$

i.e. $a_+\psi_n$ is an eigenstate with energy

$$E^* = E_n + \hbar\omega.$$

Similarly, $a_-\psi_n = (E_n - \hbar\omega)\psi_n$,

and $a_-a_-\psi_n = (E_n - 2\hbar\omega)\psi_n$, etc.

But we know (HW-2), that $E_n \geq 0\ \left(\frac{\hbar\omega}{2}\right)$.

Therefore, $a_-^k \psi_n(\xi)$ is not a normalizable eigenstate for k such that $E_n - k\hbar\omega < 0$.

There exists the lowest energy state with energy $E_0 = \frac{1}{2}\hbar\omega$,

$$a_- \psi_0(\xi) = 0 \quad \text{gives} \quad \left(\frac{d}{d\xi} + \xi\right)\psi_0(\xi) = 0$$

and $\psi_0(\xi) = \cancel{\alpha} e^{-\xi^2/2}$.

E_0 is found from $E_0 \psi_0(\xi) = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)\psi_0$

$$= \hbar\omega\left(0 + \frac{1}{2}\right)\psi_0$$

We can numerate all other states according to their energy $E_n = \hbar\omega(n + 1/2)$, ~~whic~~ The latter equation means that

$a_+ a_- \psi_n(\xi) = n\psi_n(\xi)$, and can be found as $\alpha_n a_+^n \psi_0(\xi) = \psi_n(\xi)$.

Another property of operators $a_\pm$:

$$\int_{-\infty}^{+\infty} f^*(\xi)\, \hat{a}_\pm g(\xi)\, d\xi = \int_{-\infty}^{+\infty} (a_\mp f)^* g\, d\xi$$

The proof is straitforward:

$$\int_{-\infty}^{+\infty} f^*(a_\pm g)\, d\xi = \frac{1}{\sqrt{2}} \int f^*(\xi) \cdot \left(\mp\frac{d}{d\xi} + \xi\right) g(\xi)\, d\xi$$

$$= \frac{1}{\sqrt{2}} \int \left(\pm\frac{d}{d\xi} f^* + \xi f^*\right) g(\xi)\, d\xi = \int_{-\infty}^{+\infty} (a_\mp f^*)^* g(\xi)\, d\xi$$

13.4

$$\hat{a}_+\hat{a}_-\psi_n = n\psi_n \quad \text{and} \quad a_-\psi_n = c_n\psi_{n-1}$$
with $\int \psi_n^2\, d\xi = 1$

$$c_n^{\mp 2} = \int_{-\infty}^{+\infty} (a_\mp \psi_n)(a_\mp \psi_n)\, d\xi = \int_{-\infty}^{+\infty} \psi_n\, a_\pm a_\mp \psi_n\, d\xi$$

for "−" $= n \int_{-\infty}^{+\infty} \psi_n^2(\xi)\, d\xi = n;$ or $c_n = \sqrt{n}$, i.e.

$$a_-\psi_n(\xi) = \sqrt{n}\,\psi_n(\xi).$$

for "+" $= \int_{-\infty}^{+\infty} \psi_n(\xi)\left(\hat{a}_+\hat{a}_- + 1\right)\psi_n(\xi)\, d\xi = (n+1)$

$$a_+\psi_n(\xi) = \sqrt{n+1}\,\psi_{n+1}(\xi).$$

We derived $\psi_n(\xi)$ in lectures <u>11–12</u>:

$$\psi_n(\xi) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{2^n n!}} H_n(\xi)e^{-\xi^2/2}$$

Let us check $a_+\psi_n(\xi) = \sqrt{n+1}\,\psi_{n+1}(\xi)$

$$\frac{1}{\sqrt{2}}\left(\mp\frac{d}{d\xi} + \xi\right)\cdot\left(H_n(\xi)e^{-\xi^2/2}\right) = +\frac{1}{\sqrt{2}}\left(\mp\frac{dH_n(\xi)}{d\xi}\cdot e^{-\xi^2/2}\right.$$

$$\left.\pm\xi H_n(\xi)e^{-\xi^2/2} + \xi H_n e^{-\xi^2/2}\right) = {}^{+}$$

$$= \frac{1}{\sqrt{2}}\left(\mp 2nH_{n-1}(\xi) + 2\xi H_n(\xi)\right)e^{-\xi^2/2} = \sqrt{2n}\,H_{n-1}(\xi)e^{-\xi^2/2}.$$

$$= \frac{1}{\sqrt{2}}H_{n+1}(\xi)e^{-\xi^2/2},$$ ~~compare to $\psi_{n+1}(\xi) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}$~~

B.5

Compare to

$$\psi_{n+1} = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^{n+1} n!(n+1)}} H_{n+1}(\xi) e^{-\xi^2/2}$$

$$a_+ \psi_{n+1} = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2^n n!}} H_{n+1}(\xi) e^{-\xi^2/2}$$

$$= \sqrt{n+1}\ \psi_{n+1}(\xi).$$

for $a_- \psi_n(\xi)$ we have "α" real sign in above calculations