Lecture 12.

$$-\frac{d^2\psi_n(\xi)}{d\xi^2} + \xi^2\psi_n(\xi) = K_n\psi_n(\xi)$$

We found a solution in the form

$\psi_n(\xi) = A\, h_n(\xi)\, e^{-\xi^2/2}$, where

$$h_n(\xi) = \sum_{j=0}^{\infty} a_j^{(n)} \cdot \xi^j \quad \text{with} \quad a_{j+2} = a_j \frac{2j - K_n + 1}{(j+1)(j+2)}$$

All even/odd/coefficients are connected and are fixed by a_0 / a_1/.

$$h_n(\xi) = \underbrace{\sum_{\cancel{k=0}}^{\infty} a_{2k}^{(n)} \xi^{2k}}_{\text{even powers}} + \underbrace{\sum_{k=0}^{\infty} a_{2k+1}^{(n)} \xi^{2k+1}}_{\text{odd powers.}}$$

We also found that $a_k \sim \frac{1}{k} a_{k-1} \sim \frac{1}{k!} a_{0/1}$,

therefore $h_n^{e/o}(\xi) \propto \sum_k \frac{1}{k!} \xi^{2k} = e^{\xi^2}$,

unless $a_k = 0$ for "$k \geq n$." $n \in \mathbb{Z}_{\geq 0}$.

The chain breaks if $a^n_{(k+1)} = 0$ or $\frac{2k - K_n + 1}{(k+1)(k+2)} = 0$.

$$K_n = 2n + 1 \quad \text{or} \quad E_n = \hbar\omega(n + 1/2).$$

The choice of K_n allows us to break only either even or odd sequence, we must choose a_1 or a_0 to be equal to zero to keep $h_n(\xi) < e^{\xi^2}$ (non-exponential)

12.2 If n is even

$$h_n(\xi) = \sum_{K=0}^{n/2} a_{2K}^{(n)} \xi^{2K}$$ and

$$a_2^{(n)} = a_0^{(n)} \frac{2\cdot 0+1-K_n}{1\cdot 2}$$

$$a_4^{(n)} = a_2^{(n)} \frac{2\cdot 2-K_n+1}{3\cdot 4}$$

.....

$n=0$ $K_0 = 1$

$n=1$ $K_1 = 3$ ← odd

$n=2$ $K_2 = 5$

$n=3$ $K_3 = 7$ ← odd

$$a_3^{(n)} = a_1^{(n)} \frac{2-K_n+1}{2\cdot 3}$$

$$a_5^{(n)} = a_3^{(n)} \frac{6-K_n+1}{4\cdot 5}$$

$h_n(\xi) = (-)^n h_n(-\xi)$, i.e. solutions may be symmetric or antisymmetric ~~even or odd~~ with respect to $x \to -x$ (inversion of coordinate reflection).

Polynomials $h_n(\xi)$ have ~~their~~ own name, ~~call and notation $H_n(\xi)$ is a polynomial~~

They are called the Hermite polynomials $H_n(\xi)$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) e^{-\frac{m\omega \xi^2}{2\hbar}}.$$

12.3 It turns out that $H_n(\xi)$ can be defined as

$$H_n(\xi) = (-1)^n e^{\xi^2} \left(\frac{d}{d\xi}\right)^n e^{-\xi^2}$$

There are two important properties

$$H_{n+1}(\xi) = 2\xi H_n(\xi) - 2n H_{n-1}(\xi)$$

and $$\frac{dH_n(\xi)}{d\xi} = 2n H_{n-1}(\xi)$$

Do it at home!

The wave functions $\psi_n(\xi)$ are

1. Normalized to 1 $\int_{-\infty}^{+\infty} \psi_n^2(\xi)\, d\xi = 1.$

$$\frac{1}{2^n n!} \int_{-\infty}^{+\infty} \left(e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2}\right) \cdot \frac{d^n}{d\xi^n} e^{-\xi^2}\, d\xi =$$

$$= \frac{1}{2^n n!} \int_{-\infty}^{+\infty} \left[\frac{d^n}{d\xi^n} H_n(\xi)\right] e^{-\xi^2} d\xi = \left[\prod_{k=1}^{n} (2k)\right] \frac{H_0}{2^n n!} \int_{-\infty}^{+\infty} e^{-\xi^2} d\xi$$

$$= \sqrt{\pi}$$

2. $\int_{-\infty}^{+\infty} \psi_n(\xi)\psi_m(\xi)\, d\xi = 0$ for $n \neq m$

3. Complete basis (mathematics, calculus).

12.4]

Algebraic Method, "Operator"

$$\left(-\frac{d^2}{d\xi^2}+\xi^2\right)\psi_n(\xi) = K_n\,\psi_n(\xi)$$

$$a_{+} = \left(+\frac{d}{d\xi}+\xi\right)\frac{1}{\sqrt{2}} \qquad a_{+} = \left(-\frac{d}{d\xi}+\xi\right)\frac{1}{\sqrt{2}}$$

$$a_+ a_-\,\psi_n(\xi) = \left(+\frac{d}{d\xi}-\xi\right)\left(-\frac{d}{d\xi}-\xi\right)\psi_n(\xi)$$

$$= \left(+\frac{d}{d\xi}-\xi\right)\left(-\frac{d\psi_n(\xi)}{d\xi}-\xi\,\psi_n(\xi)\right)\frac{1}{2}$$

$$= \frac{1}{2}\left(-\frac{d^2\psi_n(\xi)}{d\xi^2} - \psi_n(\xi) - \cancel{\xi\frac{d\psi_n}{d\xi}} + \cancel{\xi\frac{d\psi_n(\xi)}{d\xi}} + \xi^2\psi_n(\xi)\right)$$

$$= \left[-\frac{d^2\psi_n(\xi)}{d\xi^2} + \xi^2\psi_n(\xi) - \psi_n(\xi)\right]\cdot\frac{1}{2}$$

(**) Evaluate $a_-a_+\psi_n(\xi)$ at home.

$$\left(2a_+a_- + 1\right)\psi_n(\xi) = K_n\,\psi_n(\xi)$$

h.w. $\left(2a_+a_- + 1\right)\psi_n(\xi) = 2E_n\,\psi_n(\xi)$

Repeat above with $a_-a_+\psi$, we obtain

$$a_-a_+ - a_+a_- = 1.$$

12.5

Consider $f(\xi) = a_+ \psi_n(\xi)$, where

$$\hbar\omega\left(a_+a_- + \frac{1}{2}\right) = E_n \psi_n.$$

$$\hbar\omega\left(a_+a_- + \frac{1}{2}\right) f(\xi) = \hbar\omega\left(a_+a_-a_+ + \frac{1}{2}a_+\right)\psi_n$$

$$= \hbar\omega\, a_+\left[a_+a_- + \left(\frac{1}{2}+1\right)a_+\right]\psi_n$$

$$= \left(E_n + \hbar\omega\right) a_+ \psi_n(\xi) \; ; \qquad E_{n+1} = E_n + \hbar\omega.$$

We identify $a_+ \psi_n(\xi)$ as $\alpha \psi_{n+1}(\xi)$,

α conserve normalization, but it is also a solution with energy $E_n + \hbar\omega$.

Similarly, $f(\xi) = a_- \psi_n(\xi)$ corresponds to an eigen state with $E_n \to E_n - \hbar\omega$

But energy $E_n > 0$ ($\frac{\hbar\omega}{2}$ from HW).

$a_-^k \psi_n(\xi)$ does not exist for $k > \frac{E_n}{\hbar\omega}$

$$\left(\frac{d}{d\xi} + \xi\right)\psi_0(\xi) = 0 \text{ gives } \quad \psi_0(\xi) \propto e^{-\xi^2/2},$$

and corresponds to our original $\psi_0(\xi)$.

2.7

$$H_{n+1} = (-1)^{n+1} e^{\xi^2} \frac{d}{d\xi} \frac{d^n}{d\xi^n} e^{-\xi^2}$$

$$= (-1)^{n+1} e^{\xi^2} \frac{d^n}{d\xi^n} \left[(-2\xi) e^{-\xi^2} \right]$$

$$= (-1)^{n} e^{\xi^2} \left(\frac{d^n}{d\xi^n} e^{-\xi^2} \right) \cdot 2\xi$$

$$+ (-1)^{n} e^{\xi^2} \left(\frac{d^{n-1}}{d\xi^{n-1}} e^{-\xi^2} \right) 2n \frac{d}{d\xi} \xi$$

$$= 2\xi H_n(\xi) - 2n H_{n-1}(\xi) \Rightarrow \boxed{H_{n+1}(\xi) = 2\xi H_n(\xi) - 2n H_{n-1}(\xi)}$$

$$\frac{dH_n}{d\xi} = (-1)^n \frac{d}{d\xi} \left[e^{\xi^2} \left(\frac{d^n}{d\xi^n} e^{-\xi^2} \right) \right] =$$

$$= (-1)^n 2\xi e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2}$$

$$+ (-1)^n e^{\xi^2} \frac{d^{n+1}}{d\xi^{n+1}} e^{-\xi^2} =$$

$$= 2\xi H_n(\xi) - H_{n+1}(\xi) = 2n H_{n-1}(\xi),$$

$$\boxed{\frac{dH_n(\xi)}{d\xi} = 2n H_{n-1}(\xi).}$$