Quantum Mechanics, Physics 531
Midterm Exam 1, March 12, 2008

Consider a harmonic oscillator, described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(x), \quad U(x) = \frac{1}{2}m\omega^2 x^2.$$

The wave function at $t = 0$ is given by

$$\psi(x/\lambda_q, t = 0) = \psi(\xi, t = 0) = A(\xi - 1)\exp(-\xi^2/2), \quad \lambda_q = \sqrt{\frac{\hbar}{m\omega}}.$$

Question 1 (10 points).
Find the representation of $\psi(x/\lambda_q, t = 0)$ in the basis of eigen states of the Hamiltonian, see Eq. (1) below. Calculate the normalization factor A.

Question 2. (10 points).
Write down the wave function $\psi(x/\lambda_q, t)$ at arbitrary time t.

Question 3. (15 points).
What is the average value of energy of the oscillator in state $|\psi(x,t)\rangle$? If the energy of an oscillator were measured, what values the measurement may give and with what probabilities?

Question 4. (20 points).
Calculate the average values of the coordinate, $\langle x(t)\rangle = \langle\psi(x,t)|\hat{x}|\psi(x,t)\rangle$, and the momentum, $\langle p(t)\rangle = \langle\psi(x,t)|\hat{p}|\psi(x,t)\rangle$.

Question 5. (20 points).
Evaluate $\langle\psi(x,t)|\hat{x}^2|\psi(x,t)\rangle$ and $\langle\psi(x,t)|\hat{p}^2|\psi(x,t)\rangle$.

Question 6. (15 points).
Check that the result of the previous question is consistent with the Ehrenfest's theorem:

$$\frac{d\langle x(t)\rangle}{dt} = \frac{\langle p(t)\rangle}{m}, \quad \frac{d\langle p(t)\rangle}{dt} = \left\langle -\frac{\partial U(x)}{\partial x}\right\rangle = -m\omega^2\langle x(t)\rangle.$$

Supplementary information: The first few eigen functions of the harmonic oscillator are given by

$$\begin{aligned} \psi_0(\xi) &= \frac{1}{\sqrt[4]{\pi}}\exp(-\xi^2/2); \\ \psi_1(\xi) &= \frac{\sqrt{2}}{\sqrt[4]{\pi}}\xi\exp(-\xi^2/2); \\ \psi_2(\xi) &= \frac{2\xi^2 - 1}{\sqrt{2}\sqrt[4]{\pi}}\exp(-\xi^2/2) \end{aligned} \tag{1}$$

You may need to use the integrals in the form

$$I_n = \int_{-\infty}^{+\infty} \xi^n \exp(-\xi^2)d\xi, \tag{2}$$

which have the following values $I_0 = \sqrt{\pi}$, $I_1 = I_3 = I_5 = 0$, $I_2 = \sqrt{\pi}/2$, $I_4 = 3\sqrt{\pi}/4$ and $I_6 = 15\sqrt{\pi}/8$.

Consider a wave function

$$\psi\left(\frac{x}{\lambda_q}\right) = \psi(\xi) = A(1+\xi)e^{-\xi^2/2},$$

where $\lambda_q = \sqrt{\frac{\hbar}{m\omega}}$.

$$C_1 = \frac{+\sqrt[4]{\pi}}{\sqrt{2}} A \qquad C_0 = \sqrt[4]{\pi} A$$

$$C_0^2 + C_1^2 = 1 \qquad \frac{\sqrt{\pi}}{2} A^2 + \sqrt{\pi} A^2 = 1; \quad A^2 = \frac{2}{3\sqrt{\pi}}$$

$$A = \sqrt{\frac{2}{3\sqrt{\pi}}}$$

We obtain the following normalized function $\psi\left(\frac{x}{\lambda_q}\right) = \psi(\xi) = \sqrt{\frac{2}{3\sqrt{\pi}}}(1+\xi)e^{-\xi^2/2}$

$$C_0 = \sqrt{\frac{2}{3}} \qquad C_1 = \sqrt{\frac{1}{3}}.$$

Evolution of the wave function in time can be written as

$$\psi(x,t) = C_0 \psi_0(x) e^{-iE_0 t/\hbar} + C_1 \psi_1(x) e^{-iE_1 t/\hbar}$$

with $E_0 = \hbar\omega/2$ and $E_1 = \frac{3\hbar\omega}{2}$

$$\psi(x,t) = \sqrt{\frac{2}{3}}\, \psi_0(x) e^{-i\omega t/2}$$

$$+ \frac{1}{\sqrt{3}}\, \psi_1(x)\, e^{-3i\omega t/2}$$

To calculate $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$,

we use the representation

for $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}_+$ and $\hat{a}_-$:

$$x = \frac{1}{\sqrt{2}} \lambda_q \left(\hat{a}_+ + \hat{a}_-\right) \qquad \hat{p} = \frac{i}{\sqrt{2}} \frac{\hbar}{\lambda_q} (a_+ - a_-),$$

and use the following property of

a_+ (a_-):

$$\hat{a}_+ \psi_n\left(\frac{x}{\lambda_q}\right) = \sqrt{n+1}\; \psi_{n+1}\left(\frac{x}{\lambda_q}\right)$$

$$\hat{a}_- \psi_n\left(\frac{x}{\lambda_q}\right) = \sqrt{n}\; \psi_{n-1}\left(\frac{x}{\lambda_q}\right)$$

$$\langle x \rangle = \langle \psi(x,t) | \left(\hat{a}_+ + \hat{a}_-\right) | \psi(x,t) \rangle \cdot \frac{\lambda_q}{\sqrt{2}}$$

$$|\psi(x,t)\rangle = \frac{\sqrt{2}}{\sqrt{3}} e^{-i\omega t/2} |0\rangle + \frac{1}{\sqrt{3}} |1\rangle e^{-3i\omega t/2}$$

$$(\hat{a}_+ + \hat{a}_-)|\psi(x,t)\rangle = \sqrt{\frac{2}{3}}\, e^{-i\omega t/2}|1\rangle + \frac{\sqrt{2}}{\sqrt{3}} e^{-\frac{3i\omega t}{2}}|2\rangle$$

$$+ 0 + \frac{1}{\sqrt{3}}|0\rangle e^{-\frac{3i\omega t}{2}}$$

$$\langle\psi(x,t)|(\hat{a}_+ + \hat{a}_-)|\psi(x,t)\rangle =$$

$$= \left(\sqrt{\frac{2}{3}}\langle 0| e^{+i\omega t/2} + \frac{1}{\sqrt{3}}\langle 1| e^{+\frac{3i\omega t}{2}}\right) \times (\dots)$$

$$= \frac{\sqrt{2}}{3} e^{-\frac{2i\omega t}{2}} + \frac{1}{\sqrt{3}}\sqrt{\frac{2}{3}}\, e^{i\omega t} = \frac{\sqrt{2}}{3}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$= \frac{\sqrt{2}}{3}\cdot 2\cos\omega t$$

$$\langle x\rangle = \frac{2\lambda_q}{3}\cos\omega t$$

$$\langle p\rangle = \langle\psi(x,t)|(\hat{a}_+ - \hat{a}_-)|\psi(x,t)\rangle \frac{i\hbar}{\sqrt{2}\lambda_q}$$

$$= \frac{i\hbar}{\sqrt{2}\lambda_q}\langle\psi(x,t)|\left(\sqrt{\frac{2}{3}}\, e^{-i\omega t/2}|1\rangle + \sqrt{\frac{2}{3}}\, e^{-\frac{3i\omega t}{2}}|2\rangle - \frac{1}{\sqrt{3}}|0\rangle e^{-3i\omega t}\right)$$

$$= \frac{i\hbar}{\sqrt{2}\lambda_q}\left(\frac{\sqrt{2}}{3} e^{i\omega t} - \frac{\sqrt{2}}{3} e^{-i\omega t}\right) = \frac{2i\hbar}{3\lambda_q}\, i\sin\omega t = -\frac{2\hbar}{3\lambda_q}\sin\omega t.$$

$\langle x^2 \rangle$ and $\langle p^2 \rangle$

$$\hat{x}^2 = \frac{\lambda_q^2}{2} (\hat{a}_+ + \hat{a}_-)^2$$

see above

$$\hat{x}^2 |\psi(x,t)\rangle = \frac{\lambda_q^2}{2} (a_+ + a_-) \overrightarrow{(a_+ + a_-)|\psi(x,t)\rangle}$$

$$= \frac{\lambda_q^2}{2} \cdot \Big\{ \sqrt{\frac{2 \cdot 2}{3}} e^{-i\omega t/2} |2\rangle + \ldots |3\rangle + \frac{1}{\sqrt{3}} e^{-3i\omega t/2} |1\rangle$$

$$+ \sqrt{\frac{2}{3}} e^{-i\omega t/2} |0\rangle + \sqrt{\frac{2 \cdot 1}{3}} e^{-3i\omega t/2} |1\rangle + 0 \Big\}$$

$$\langle x^2 \rangle = \Big\{ \frac{2}{3} + \frac{\sqrt{2 \cdot 2}}{3} \Big\} \frac{1}{3} \Big\} \cdot \frac{\lambda_q^2}{2} = \frac{\lambda_q^2 5}{6}$$

$$\langle p^2 \rangle = \frac{\hbar^2}{2\lambda_q^2} \langle (\hat{a}_+ - \hat{a}_-) \psi(x,t) | (a_+ - a_-) \psi(x,t) \rangle$$

$$= \frac{\hbar^2}{2\lambda_q^2} \left(\frac{2}{3} + \frac{2}{3} + \frac{1}{3} \right) = \frac{5}{6} \frac{\hbar^2}{\lambda_q^2}$$

Average energy $\bar{E} = \dfrac{\langle p^2 \rangle}{2m} + \dfrac{m\omega^2 \langle x^2 \rangle}{2}$

$$= \frac{5}{6} \cdot \frac{\hbar^2}{(\hbar/m\omega) 2m} + \frac{m\omega^2 \hbar}{2m\omega} \cdot \frac{5}{6} = \frac{5}{12} \cdot \hbar\omega \cdot 2 = \frac{5}{6} \hbar\omega$$

Alternatively, $E = \left(\frac{1}{2}\cdot\frac{2}{3} + \frac{3}{2}\cdot\frac{1}{3}\right)\hbar\omega$

$$= \frac{5}{6}\hbar\omega.$$

But each measurement gives

$\frac{1}{2}\hbar\omega$ with probability $\frac{2}{3}$

$\frac{3}{2}\hbar\omega$ with probability $\frac{1}{3}$.

Calculate $\frac{d}{dt}\langle \dot{x}\rangle = -\frac{2\lambda_q\omega}{3}\cdot\sin\omega t$

$$= -\frac{2}{3}\sin\omega t\cdot\sqrt{\frac{\hbar\omega}{m}} =$$

$$= -\frac{2}{3}\frac{\hbar}{m}\sqrt{\frac{m\omega}{\hbar}}\sin\omega t$$

$$= +\frac{\langle p\rangle}{m}$$

Similarly, $\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle = -m\omega\langle x\rangle$,

Indeed $\frac{d\langle p\rangle}{dt} = -\frac{2}{3}\frac{\hbar}{\lambda_q}\cdot\omega\cos\omega t =$

$$= -\frac{2}{3}\lambda_q\cos\omega t\cdot\frac{\hbar}{\lambda_q^2} = -\langle x(t)\rangle\cdot m\omega.$$

$$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \left(\frac{5}{6} - \frac{4}{9}\sin^2\omega t\right)\frac{\hbar^2}{\lambda_g^2}$$

$$= \left(\frac{15 - 8\sin^2\omega t}{18}\right)\frac{\hbar^2}{\lambda_g^2} = \frac{7}{18}\frac{\hbar^2}{\lambda_g^2} + \frac{8}{18}\cos^2\omega t\,\frac{\hbar^2}{\lambda_g^2}$$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \left(\frac{5}{6} - \frac{4}{9}\cos^2\omega t\right)\lambda_g^2$$

$$= \frac{7}{18}\lambda_g^2 + \frac{8}{18}\sin^2\omega t\,\lambda_g^2$$

$$\sigma_x^2\cdot\sigma_p^2 = \hbar^2\left(\frac{7}{18} + \frac{8}{18}\cos^2\omega t\right)\left(\frac{7}{18} + \frac{8}{18}\sin^2\omega t\right)$$

$$= \hbar^2\left[\left(\frac{7}{18}\right)^2 + \frac{7\cdot 8}{(18)^2}\left(\cos^2\omega t + \sin^2\omega t\right) + \frac{64}{18^2}\cos^2\omega t\sin^2\omega t\right]$$

$$= \hbar^2\left[\frac{49+56}{18^2} + \frac{16}{18^2}\cdot\sin^2 2\omega t\right] \geq \frac{\hbar^2}{4}.$$

$$\sigma_x\cdot\sigma_E \geq \frac{\hbar}{2m}|\langle p\rangle|$$