Quantum Mechanics, Physics 531
Homework Assignment 7, due May 9, 2008

Problem 1. Problem 9.1.

Problem 2. Problem 9.15.

Problem 3. Problem 10.9.

Problem 4. Problem 29 in Chapter 5 of Sakurai.

$\left(-\frac{iE_b}{\hbar}\right) e^{-iE_bt/\hbar}\Big].$

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${}_be^{-iE_bt/\hbar}\Big].$ [9.8]

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***Problem 9.1** A hydrogen atom is placed in a (time-dependent) electric field $\mathbf{E} = E(t)\hat{k}$. Calculate all four matrix elements H'_{ij} of the perturbation $H' = eEz$ between the ground state ($n = 1$) and the (quadruply degenerate) first excited states ($n = 2$). Also show that $H'_{ii} = 0$ for all five states. *Note:* There is only one integral to be done here, if you exploit oddness with respect to z; only one of the $n = 2$ states is "accessible" from the ground state by a perturbation of this form, and therefore the system functions as a two-state configuration—assuming transitions to higher excited states can be ignored.

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gure 9.6. Note that the
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is indeed much longer
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igly) called **forbidden**

9.74. *Hint:* First show

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****Problem 9.15** Develop time-dependent perturbation theory for a multilevel system, starting with the generalization of Equations 9.1 and 9.2:

$$H_0\psi_n = E_n\psi_n, \quad \langle\psi_n|\psi_m\rangle = \delta_{nm}. \tag{9.79}$$

At time $t = 0$ we turn on a perturbation $H'(t)$, so that the total Hamiltonian is

$$H = H_0 + H'(t). \tag{9.80}$$

(a) Generalize Equation 9.6 to read

$$\Psi(t) = \sum c_n(t)\psi_n e^{-iE_nt/\hbar}, \tag{9.81}$$

and show that

$$\dot{c}_m = -\frac{i}{\hbar}\sum_n c_n H'_{mn} e^{i(E_m-E_n)t/\hbar}, \tag{9.82}$$

where

$$H'_{mn} \equiv \langle\psi_m|H'|\psi_n\rangle. \tag{9.83}$$

(b) If the system starts out in the state ψ_N, show that (in first-order perturbation theory)

$$c_N(t) \cong 1 - \frac{i}{\hbar}\int_0^t H'_{NN}(t')\,dt'. \tag{9.84}$$

and

$$c_m(t) \cong -\frac{i}{\hbar}\int_0^t H'_{mN}(t')e^{i(E_m-E_N)t'/\hbar}\,dt', \quad (m \neq N). \tag{9.85}$$

(c) For example, suppose H' is *constant* (except that it was turned on at $t = 0$, and switched off again at some later time t). Find the probability of transition from state N to state M ($M \neq N$), as a function of t. *Answer:*

$$4|H'_{MN}|^2\frac{\sin^2[(E_N - E_M)t/2\hbar]}{(E_N - E_M)^2}. \tag{9.86}$$

(d) Now suppose H' is a sinusoidal function of time: $H' = V\cos(\omega t)$. Making the usual assumptions, show that transitions occur only to states with energy $E_M = E_N \pm \hbar\omega$, and the transition probability is

$$P_{N\to M} = |V_{MN}|^2\frac{\sin^2[(E_N - E_M \pm \hbar\omega)t/2\hbar]}{(E_N - E_M \pm \hbar\omega)^2}. \tag{9.87}$$

(e) Suppose a multilevel system is immersed in incoherent electromagnetic radiation. Using Section 9.2.3 as a guide, show that the transition rate for stimulated emission is given by the same formula (Equation 9.47) as for a two-level system.

***Problem 10.9 Suppose the one-dimensional harmonic oscillator (mass m, frequency ω) is subjected to a driving force of the form $F(t) = m\omega^2 f(t)$, where $f(t)$ is some specified function (I have factored out $m\omega^2$ for notational convenience; $f(t)$ has the dimensions of *length*). The Hamiltonian is

$$H(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 - m\omega^2 x f(t). \qquad [10.90]$$

Assume that the force was first turned on at time $t = 0$: $f(t) = 0$ for $t \leq 0$. This system can be solved exactly, both in classical mechanics and in quantum mechanics.[21]

(a) Determine the *classical* position of the oscillator, assuming it started from rest at the origin ($x_c(0) = \dot{x}_c(0) = 0$). *Answer:*

$$x_c(t) = \omega \int_0^t f(t') \sin[\omega(t - t')]\, dt'. \qquad [10.91]$$

(b) Show that the solution to the (time-dependent) Schrödinger equation for this oscillator, assuming it started out in the nth state of the *undriven* oscillator ($\Psi(x, 0) = \psi_n(x)$ where $\psi_n(x)$ is given by Equation 2.61), can be written as

$$\Psi(x, t) = \psi_n(x - x_c)e^{\frac{i}{\hbar}\left[-(n+\frac{1}{2})\hbar\omega t + m\dot{x}_c(x - \frac{x_c}{2}) + \frac{m\omega^2}{2}\int_0^t f(t')x_c(t')dt'\right]}. \qquad [10.92]$$

(c) Show that the eigenfunctions and eigenvalues of $H(t)$ are

$$\psi_n(x, t) = \psi_n(x - f); \quad E_n(t) = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{1}{2}m\omega^2 f^2. \qquad [10.93]$$

(d) Show that in the adiabatic approximation the classical position (Equation 10.91) reduces to $x_c(t) \cong f(t)$. State the precise criterion for adiabaticity, in this context, as a constraint on the time derivative of f. ***Hint:*** Write $\sin[\omega(t - t')]$ as $(1/\omega)(d/dt')\cos[\omega(t - t')]$ and use integration by parts.

(e) Confirm the adiabatic theorem for this example, by using the results in (c) and (d) to show that

$$\Psi(x, t) \cong \psi_n(x, t)e^{i\theta_n(t)}e^{i\gamma_n(t)}. \qquad [10.94]$$

Check that the dynamic phase has the correct form (Equation 10.39). Is the geometric phase what you would expect?

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9. Consider a composite system made up of two spin $\frac{1}{2}$ objects. For $t < 0$, the Hamiltonian does not depend on spin and can be taken to be zero by suitably adjusting the energy scale. For $t > 0$, the Hamiltonian is given by

$$H = \left(\frac{4\Delta}{\hbar^2}\right)\mathbf{S}_1 \cdot \mathbf{S}_2.$$

Suppose the system is in $|+-\rangle$ for $t \leq 0$. Find, as a function of time, the probability for being found in each of the following states $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$:

a. By solving the problem exactly.
b. By solving the problem assuming the validity of first-order time-dependent perturbation theory with H as a perturbation switched on at $t = 0$. Under what condition does (b) give the correct results?