Quantum Mechanics, Physics 531
Homework Assignment 7, due May 9, 2008

Problem 1. Problem 9.1.

Problem 2. Problem 9.15.


Problem 4. Problem 29 in Chapter 5 of Sakurai.
Problem 9.1 A hydrogen atom is placed in a (time-dependent) electric field \( E = E(t) \hat{k} \). Calculate all four matrix elements \( H_{ij}^{'} \) of the perturbation \( H' = eEz \) between the ground state \((n = 1)\) and the (quadruply degenerate) first excited states \((n = 2)\). Also show that \( H_{ii}^{'} = 0 \) for all five states. Note: There is only one integral to be done here, if you exploit oddness with respect to \( z \); only one of the \( n = 2 \) states is "accessible" from the ground state by a perturbation of this form, and therefore the system functions as a two-state configuration—assuming transitions to higher excited states can be ignored.

Problem 9.15 Develop time-dependent perturbation theory for a multilevel system, starting with the generalization of Equations 9.1 and 9.2:

\[
H_0 \psi_n = E_n \psi_n, \quad (\psi_n | \psi_m) = \delta_{nm}. \tag{9.79}
\]

At time \( t = 0 \) we turn on a perturbation \( H'(t) \), so that the total Hamiltonian is

\[
H = H_0 + H'(t). \tag{9.80}
\]

(a) Generalize Equation 9.6 to read

\[
\psi(t) = \sum c_n(t) \psi_n e^{-iE_n t/\hbar}, \tag{9.81}
\]

and show that

\[
c_m = -\frac{i}{\hbar} \sum_n c_n H_{mn}' e^{i(E_m - E_n)t/\hbar}, \tag{9.82}
\]

where

\[
H_{mn}' = (\psi_m | H'(t) | \psi_n). \tag{9.83}
\]

(b) If the system starts out in the state \( \psi_N \), show that (in first-order perturbation theory)

\[
c_N(t) \equiv 1 - \frac{i}{\hbar} \int_0^t H_{NN}'(t') dt'. \tag{9.84}
\]

and

\[
c_m(t) \equiv -\frac{i}{\hbar} \int_0^t H_{mn}'(t') e^{i(E_m - E_N)t'/\hbar} dt', \quad (m \neq N). \tag{9.85}
\]

(c) For example, suppose \( H' \) is constant (except that it was turned on at \( t = 0 \), and switched off again at some later time \( t \)). Find the probability of transition from state \( N \) to state \( M (M \neq N) \), as a function of \( t \). Answer:

\[
4|H_{MN}'|^2 \sin^2 \left( \frac{(E_N - E_M)t}{2\hbar} \right), \tag{9.86}
\]

(d) Now suppose \( H' \) is a sinusoidal function of time: \( H' = V \cos(\omega t) \). Making the usual assumptions, show that transitions occur only to states with energy \( E_M = E_N \pm \hbar \omega \), and the transition probability is

\[
P_{N \rightarrow M} = |V_{MN}|^2 \sin^2 \left( \frac{(E_N - E_M \pm \hbar \omega)t}{2\hbar} \right) \frac{(E_N - E_M \pm \hbar \omega)^2}{(E_N - E_M)^2}. \tag{9.87}
\]

(e) Suppose a multilevel system is immersed in incoherent electromagnetic radiation. Using Section 9.2.3 as a guide, show that the transition rate for stimulated emission is given by the same formula (Equation 9.47) as for a two-level system.
**Problem 10.9** Suppose the one-dimensional harmonic oscillator (mass \( m \), frequency \( \omega \)) is subjected to a driving force of the form \( F(t) = m\omega^2 f(t) \), where \( f(t) \) is some specified function (I have factored out \( m\omega^2 \) for notational convenience; \( f(t) \) has the dimensions of length). The Hamiltonian is

\[
H(t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 - m\omega^2 x f(t). \tag{10.90}
\]

Assume that the force was first turned on at time \( t = 0 \): \( f(t) = 0 \) for \( t \leq 0 \). This system can be solved exactly, both in classical mechanics and in quantum mechanics. \(^{21}\)

(a) Determine the classical position of the oscillator, assuming it started from rest at the origin \( (x_0(0) = \dot{x}_0(0) = 0) \). \textit{Answer:}

\[
x_c(t) = \omega \int_0^t f(t') \sin(\omega(t - t')) \, dt'. \tag{10.91}
\]

(b) Show that the solution to the (time-dependent) Schrödinger equation for this oscillator, assuming it started out in the \( n \)th state of the undriven oscillator \( \psi_n(x) \) where \( \psi_n(x) \) is given by Equation 2.61, can be written as

\[
\psi(x, t) = \psi_n(x - x_c) e^{i \left[ -\frac{1}{2} (n + \frac{1}{2}) \hbar \omega t + mx_c(x - \frac{n}{2}) + \frac{m\omega^2}{2} \int_0^t f(t') x_c(t') \, dt' \right]} \tag{10.92}
\]

(c) Show that the eigenfunctions and eigenvalues of \( H(t) \) are

\[
\psi_n(x, t) = \psi_n(x - f); \quad E_n(t) = \left( n + \frac{1}{2} \right) \hbar \omega - \frac{1}{2} m\omega^2 f^2. \tag{10.93}
\]

(d) Show that in the adiabatic approximation the classical position (Equation 10.91) reduces to \( x_c(t) \cong f(t) \). State the precise criterion for adiabaticity, in this context, as a constraint on the time derivative of \( f \). \textit{Hint:} Write \( \sin[\omega(t - t')] \) as \( (1/\omega)(d/dt') \cos[\omega(t - t')] \) and use integration by parts.

(e) Confirm the adiabatic theorem for this example, by using the results in (c) and (d) to show that

\[
\psi(x, t) \cong \psi_n(x, t) e^{i \dot{\theta}_n(t)} e^{i \hbar \omega(t)}. \tag{10.94}
\]

Check that the dynamic phase has the correct form (Equation 10.39). Is the geometric phase what you would expect?

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9. Consider a composite system made up of two spin \( \frac{1}{2} \) objects. For \( t < 0 \), the Hamiltonian does not depend on spin and can be taken to be zero by suitably adjusting the energy scale. For \( t > 0 \), the Hamiltonian is given by

\[
H = \left( \frac{4\Delta}{\hbar^2} \right) S_1 \cdot S_2.
\]

Suppose the system is in \( |+ - \rangle \) for \( t \leq 0 \). Find, as a function of time, the probability for being found in each of the following states \( |++ \rangle \), \( |+ - \rangle \), \( |- + \rangle \), and \( |-- \rangle \):

a. By solving the problem exactly.

b. By solving the problem assuming the validity of first-order time-dependent perturbation theory with \( H \) as a perturbation switched on at \( t = 0 \). Under what condition does (b) give the correct results?