Quantum Mechanics, Physics 531
Homework Assignment 6, due April 28, 2008

Problem 1. Problem 6.2.

Problem 2. Problem 6.5. Using the result of the second order perturbation, calculate the electric field susceptibility of a harmonic oscillator.

Problem 3. Problem 6.9.

Problem 4. Problem 6.14.

Problem 5. Problem 6.23.

Problem 6. Problem 6.32.

Problem 7. Problem 11 in Chapter 5 of Sakurai.

Problem 8. Problem 17(a) in Chapter 5 of Sakurai; part (b) is not required.

***Problem 6.2** For the harmonic oscillator $[V(x) = (1/2)kx^2]$, the allowed energies are

$$E_n = (n + 1/2)\hbar\omega, \quad (n = 0, 1, 2, \ldots),$$

where $\omega = \sqrt{k/m}$ is the classical frequency. Now suppose the spring constant increases slightly: $k \to (1 + \epsilon)k$. (Perhaps we cool the spring, so it becomes less flexible.)

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(a) Find the *exact* new energies (trivial, in this case). Expand your formula as a power series in ϵ, up to second order.

(b) Now calculate the first-order perturbation in the energy, using Equation 6.9. What is H' here? Compare your result with part (a). *Hint:* It is not necessary—in fact, it is not *permitted*—to calculate a single integral in doing this problem.

[6.12]

nal), so

$$)\langle\psi_n^0|H'|\psi_m^0\rangle \over - E_m^0,$$

****Problem 6.5** Consider a charged particle in the one-dimensional harmonic oscillator potential. Suppose we turn on a weak electric field (E), so that the potential energy is shifted by an amount $H' = -qEx$.

(a) Show that there is no first-order change in the energy levels, and calculate the second-order correction. *Hint:* See Problem 3.33.

(b) The Schrödinger equation can be solved directly in this case, by a change of variables: $x' \equiv x - (qE/m\omega^2)$. Find the exact energies, and show that they are consistent with the perturbation theory approximation.

∗Problem 6.9 Consider a quantum system with just *three* linearly independent states. Suppose the Hamiltonian, in matrix form, is

$$\mathbf{H} = V_0 \begin{pmatrix} (1-\epsilon) & 0 & 0 \\ 0 & 1 & \epsilon \\ 0 & \epsilon & 2 \end{pmatrix},$$

where V_0 is a constant, and ϵ is some small number ($\epsilon \ll 1$).

(a) Write down the eigenvectors and eigenvalues of the *unperturbed* Hamiltonian ($\epsilon = 0$).

(b) Solve for the *exact* eigenvalues of $\mathbf{H}$. Expand each of them as a power series in ϵ, up to second order.

(c) Use first- and second-order *non*degenerate perturbation theory to find the approximate eigenvalue for the state that grows out of the nondegenerate eigenvector of H^0. Compare the exact result, from (a).

(d) Use *degenerate* perturbation theory to find the first-order correction to the two initially degenerate eigenvalues. Compare the exact results.

∗∗Problem 6.14 Find the (lowest-order) relativistic correction to the energy levels of the one-dimensional harmonic oscillator. *Hint:* Use the technique in Example 2.5.

∗∗Problem 6.23 Consider the (eight) $n = 2$ states, $|2\,l\,m_l\,m_s\rangle$. Find the energy of each state, under strong-field Zeeman splitting. Express each answer as the sum of three terms: the Bohr energy, the fine-structure (proportional to α^2), and the Zeeman contribution (proportional to $\mu_B B_{\text{ext}}$). If you ignore fine structure altogether, how many distinct levels are there, and what are their degeneracies?

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****Problem 6.32** Suppose the Hamiltonian H, for a particular quantum system, is a function of some parameter λ; let $E_n(\lambda)$ and $\psi_n(\lambda)$ be the eigenvalues and eigenfunctions of $H(\lambda)$. The **Feynman-Hellmann theorem**[22] states that

$$\frac{\partial E_n}{\partial \lambda} = \left\langle \psi_n \left| \frac{\partial H}{\partial \lambda} \right| \psi_n \right\rangle \qquad [6.103]$$

(assuming either that E_n is nondegenerate, or—if degenerate—that the ψ_n's are the "good" linear combinations of the degenerate eigenfunctions).

(a) Prove the Feynman-Hellmann theorem. *Hint:* Use Equation 6.9.

(b) Apply it to the one-dimensional harmonic oscillator, (i) using $\lambda = \omega$ (this yields a formula for the expectation value of V), (ii) using $\lambda = \hbar$ (this yields $\langle T \rangle$), and (iii) using $\lambda = m$ (this yields a relation between $\langle T \rangle$ and $\langle V \rangle$). Compare your answers to Problem 2.12, and the virial theorem predictions (Problem 3.31).

11. The Hamiltonian matrix for a two-state system can be written as

$$\mathscr{H} = \begin{pmatrix} E_1^0 & \lambda\Delta \\ \lambda\Delta & E_2^0 \end{pmatrix}.$$

Clearly the energy eigenfunctions for the unperturbed problems ($\lambda = 0$) are given by

$$\phi_1^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \phi_2^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

a. Solve this problem *exactly* to find the energy eigenfunctions ψ_1 and ψ_2 and the energy eigenvalues E_1 and E_2.

b. Assuming that $\lambda|\Delta| \ll |E_1^0 - E_2^0|$, solve the same problem using time-independent perturbation theory up to first order in the energy eigenfunctions and up to second order in the energy eigenvalues. Compare with the exact results obtained in (a).

c. Suppose the two unperturbed energies are "almost degenerate," that is,

$$|E_1^0 - E_2^0| \ll \lambda|\Delta|.$$

Show that the exact results obtained in (a) closely resemble what you would expect by applying *degenerate* perturbation theory to this problem with E_1^0 set exactly equal to E_2^0.

17. a. Suppose the Hamiltonian of a rigid rotator in a magnetic field perpendicular to the axis is of the form (Merzbacher 1970, Problem 17-1)

$$AL^2 + BL_z + CL_y$$

if terms quadratic in the field are neglected. Assuming $B \gg C$, use perturbation theory to lowest nonvanishing order to get approximate energy eigenvalues.

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