Quantum Mechanics, Physics 531
Final Exam, due May 17, 2008

Problem 1. *(20 points)* a) Prove that the expectation value of the momentum operator $\hat{p}_x = -i\hbar\partial/\partial_x$ in a stationary state $\psi(x)$ is zero, $\langle\psi|\hat{p}_x|\psi\rangle = 0$.

b) Prove that the expectation value of $\hat{p}_x^2$ in a stationary state is positive, $\langle\psi|\hat{p}_x^2|\psi\rangle > 0$.

Problem 2. *(25 points)* The Coulomb potential energy of an electron in a hydrogen atom is $V(r) = -e^2/4\pi\epsilon_0 r$, which becomes infinitely negative as $r \to 0$. What prevents the ground state energy from being infinitely negative? What happens to the ground state energy if atom contains a proton and a μ meson, the meson mass is $m_\mu \approx 206 m_e$, m_e is the electron mass.

Problem 3. *(40 points)* A particle of mass m moves in a three dimensional isotropic potential of a harmonic oscillator, $V(\mathbf{r}) = m\omega^2|\mathbf{r}|^2/2$ and is in a stationary state with energy $E = 5\hbar\omega/2$.

a) How many eigen states corresponds to this energy E? Write down these states in terms of wave functions of one dimensional harmonic oscillator in a cartesian coordinates.

b) What is the average kinetic and potential energies in any of these eigen states?

c) What is the magnitude of the total angular momentum $\hat{L}^2$ for these states?

d) Calculate the average rotational energy, $\langle\psi|\hat{L}^2/(2m|\mathbf{r}|^2)|\psi\rangle$ in any of these eigen states? Compare the result with the answer to question 3.b, explain the difference.

Problem 4. *(40 points)* The operators for spin 1/2 can be written as $\hat{S}_i = (\hbar/2)\hat{\sigma}_i$, with $i = x, y, z$ and

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The hamiltonian for such a spin in a magnetic field B along $y-$axis is $\hat{H} = -\mu B\hat{S}_y$.

a) Give the eigenvalues and the normalized eigenstates of $\hat{S}_z$ in the form

$$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{1}$$

b) Calculate the expectation value of $\hat{S}_y$ in an eigenstate of $\hat{S}_z$. Explain the result.

c) Give the eigenvalues and the normalized eigenstates of $\hat{S}_y$ in the form of Eq. (1). What are the eigenvalues and the eigenstates of the Hamiltonian $\hat{H}$?

d) For the given Hamiltonian, find the time-dependent wave function, if at $t = 0$ the wave function had the form $\psi(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Interpret the result.

see the second page!

Problem 5. *(50 points)* Consider the Jaynes-Cummings Hamiltonian, representing coupling between a harmonic oscillator and spin 1/2:

$$\hat{H} = \hbar\omega(\hat{a}^{+}\hat{a}^{-} + 1/2) + \hbar\Omega/2\hat{\sigma}_z + \lambda\left(\hat{a}^{+}\hat{\sigma}_{-} + \hat{a}^{-}\hat{\sigma}_{+}\right), \tag{2}$$

where the 2×2 matrices $\hat{\sigma}_{\pm} = (\hat{\sigma}_x \pm i\hat{\sigma}_y)/2$, and $\hat{\sigma}_{x,y}$ are defined above, the operators $\hat{a}^{\pm}$ are the raising and lowering operators of the harmonic oscillator.

a) What are the eigenvalues and eigenstates of this Hamiltonian in the absence of interaction between the oscillator and the spin, i.e. $\lambda = 0$?

b) Evaluate the expectation value of the full Hamiltonian with respect to eigenstates of non-interacting system.

c) Calculate the first order perturbation correction to the eigenstates and eigenvalues of the Hamiltonian with interaction.

d) Calculate the second order perturbation correction to the eigenvalues of the Hamiltonian with interaction.

Problem 6. *(25 points)* A particle of mass m moves in a one dimensional box with infinite potential walls,

$$V(x) = \begin{cases} 0, & |x| < a; \\ +\infty, & |x| \geq a. \end{cases}$$

a) What are the eigenfunctions and eigenenergies?

b) Can you apply the semiclassical approximation to answer question a)?

c) Make a variational estimate of the ground state energy using the trial function

$$\psi(x) = A \begin{cases} (1 - x^2/a^2), & |x| < a; \\ 0, & \text{otherwise.} \end{cases}$$